The Borel Structure of the Collections of Sub-Self-Similar Sets and Super-Self-Similar Sets

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Abstract

We show that the sets of sub-self-similar sets and super-self-similar sets are both dense, first category, F_σ subsets of $\mathcal{K}(\mathbb{R}^d)$, the Hausdorff metric space of non-empty compact, subsets of $\mathbb{R}^d$. We also investigate the set of self-similar sets as a subset of the sub-self-similar sets and the super-self-similar sets.

1 Introduction

In [Fal1], Falconer introduced the notion of sub-self-similarity as a generalization of self-similarity and showed that sub-self-similar sets retain many of the nice properties of self-similar sets. Later in [Fal2] we find the notion of a super-self-similar set. The question arises as to how strong a generalization are these new concepts. In this paper, we quantify this question using topological notions in $\mathcal{K}(\mathbb{R}^d)$, the Hausdorff metric space of non-empty compact subsets of $\mathbb{R}^d$. In particular, we show that the sets of sub-self-similar sets and super-self-similar sets are both dense, first category, F_σ subsets of $\mathcal{K}(\mathbb{R}^d)$. The fact that these sets are dense could be interpreted as meaning that we have an understanding of many compact subsets of $\mathbb{R}^d$. The fact that these sets are first category indicates that most compact sets are not encompassed in these definitions. We also consider the set of self-similar sets as a subset of the sub-self-similar sets and the super-self-similar sets. In particular, we show that the sub-self-similar sets which are not self-similar are dense in the set of sub-self-similar sets, and similarly for the super-self-similar sets. This indicates that Falconer's new concepts are a considerable generalization over the self-similar sets.

2 Definitions

We work in a fixed Euclidean space $\mathbb{R}^d$. Let $\mathcal{K}(\mathbb{R}^d)$ be the set of non-empty, compact subsets of $\mathbb{R}^d$. The Hausdorff metric ρ on $\mathcal{K}(\mathbb{R}^d)$ is defined by

$$\rho(A, B) = \max\{\sup_{x\in A}\{\text{dist}(x, B)\}, \sup_{y\in B}\{\text{dist}(y, A)\}\}.$$

A discussion of the Hausdorff metric may be found in [Ed] section 2.4. Of particular interest is theorem 2.4.4., which states that $\mathcal{K}(\mathbb{R}^d)$ is complete. This allows us to appeal to Baire category type arguments in $\mathcal{K}(\mathbb{R}^d)$. Also of note is exercise 2.4.2, which characterizes the limit of a sequence of sets in the Hausdorff metric as follows: If $A_n \to A$ in the Hausdorff metric, then

$$A = \{x : \exists \{x_n\}_{n=1}^{\infty} \text{ with } x_n \in A_n \text{ and } x_n \to x\}.$$

A function $T : \mathbb{R}^d \to \mathbb{R}^d$ is a similarity with ratio $r = r(T) > 0$ if

$$|T(x) - T(y)| = r|x - y| \ \forall x, y \in \mathbb{R}^d.$$

If $r < 1$, then T is called contractive. A fundamental result ([Ed], Thm. 4.1.3) states that if $T_i : \mathbb{R}^d \to \mathbb{R}^d$ is a contractive similarity for each $i \in \{1, \ldots, m\}$, then there is a unique, nonempty, compact set $E \subseteq \mathbb{R}^d$ such that

$$E = \cup_{i=1}^{m} T_i(E).$$

The set E is called self-similar.

Sub-self-similar sets are obtained by relaxing the equality to inclusion. Thus, the compact set E is sub-self-similar if there are contractive similarities $T_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i \in \{1, \ldots, m\}$ such that

$$E \subseteq \cup_{i=1}^{m} T_i(E).$$

Clearly any self-similar set is sub-self-similar. [Fal1] contains many other examples of sub-self-similar sets and describes their basic properties. The following lemma provides an example of a non-sub-self-similar set.

Lemma 2.1 *Let $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Then E is not a sub-self-similar set.*

Proof: Assume that $\{T_i\}_{i=1}^{m}$ are contractive similarities. We will show that $E \not\subseteq \cup_{i=1}^{m} T_i(E)$.

Suppose first that no T_i has 0 as a fixed point. Then there is a neighborhood U of 0 such that $T_i(0) \notin U$ for every $i \in \{1, \ldots, m\}$. Since 0 is the only cluster point of E, it follows that $U \cap \cup_{i=1}^{m} T_i(E)$ can contain only finitely many points. But $U \cap E$ is infinite, so $E \not\subseteq \cup_{i=1}^{m} T_i(E)$.

Now, by reordering the set $\{T_i\}_{i=1}^{m}$ if necessary, choose $n \leq m$ such that $\{T_i\}_{i=1}^{n}$ are those similarities with 0 as a fixed point. We will show that $E \setminus \cup_{i=1}^{n} T_i(E)$ is infinite. Note that $T_i(E) \cap E = \{0\}$ unless $r(T_i)$ is of the specific form $\frac{p_i}{q_i}$ where $q_i, p_i \in \mathbb{N}$ and $q_i \geq 2$. Thus if p is a prime larger than q_i for each $i \in \{1, \ldots, n\}$, then $\cup_{i=1}^{n} T_i(E)$ will contain no number of the form $\frac{1}{kp}$, where $k \in \mathbb{N}$. Now the remaining portion $\cup_{i=n+1}^{m} T_i(E)$ may contain only finitely many points of E for the reasons outlined above. Thus we again have $E \not\subseteq \cup_{i=1}^{m} T_i(E)$.□

The above argument may clearly be embedded in $\mathbb{R}^d$ by associating $\mathbb{R}$ with just one of the coordinates of $\mathbb{R}^d$. Furthermore, if E is the set in the lemma, we may obtain other non-sub-self-similar sets by scaling and translating E. Finally, the union of such a set with any finite set will be non-sub-self-similar. Using this fact together with the fact that the finite sets are dense in $\mathcal{K}(\mathbb{R}^d)$, we obtain the following important corollary.

Corollary 2.1 *The set of non-sub-self-similar sets is dense in* $\mathcal{K}(\mathbb{R}^d)$.

The super-self-similar sets were introduced in [Fal2] by reversing the inclusion. Thus, the compact set E is super-self-similar if there are contractive similarities $T_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i \in \{1, \dots, m\}$ such that

$$E \supseteq \cup_{i=1}^m T_i(E).$$

It again turns out that the super-self-similar sets retain some nice properties of the self-similar sets, although some additional assumption may need to be added (see [Fal2], cor. 3.4). As with the sub-self-similar sets, we will need the fact that the set of non-super-self-similar sets is dense in $\mathcal{K}(\mathbb{R}^d)$.

Lemma 2.2 *No finite set with more than one element is super-self-similar.*

Proof: Let F be a finite set with more than one element and let $\{T_i\}_{i=1}^m$ be contractive similarities. We will show that $F \not\supseteq \cup_{i=1}^m T_i(F)$. Let x be the fixed point of T_1 and let $y \in F$ satisfy $|x - y| = \text{dist}(x, F \setminus \{x\})$. Then clearly $T_1(y) \notin F$ so $F \not\supseteq \cup_{i=1}^m T_i(F)$.□

As the finite sets are dense in $\mathcal{K}(\mathbb{R}^d)$, we obtain the following corollary immediately.

Corollary 2.2 *The set of non-super-self-similar sets is dense in* $\mathcal{K}(\mathbb{R}^d)$.

Note that the finite sets are all sub-self-similar while the set E from Lemma 2.1 is super-self-similar for the set of transformations $\{T_1(x) = \frac{1}{2}x, T_2(x) = \frac{1}{3}x\}$.

As a notational convenience, we will denote the set of self-similar sets by ss, the set of sub-self-similar sets by sss and the set of super-self-similar sets by Sss.

3 The Main Results

In this section, we prove our main results. Theorem 3.1 states that sss is a first category, F_σ subset of $\mathcal{K}(\mathbb{R}^d)$.

Theorem 3.1 *The set of sub-self-similar sets may be expressed as the countable union of closed, nowhere dense subsets of* $\mathcal{K}(\mathbb{R}^d)$.

Proof: For $m, n \in \mathbb{N}$, define $sss_{m,n}$ to be the set of all those sub-self-similar sets E such that there exists contractive similarities $\{T_i\}_{i=1}^m$ with $\frac{1}{n} \le r(T_i) \le 1 - \frac{1}{n}$, $E \subseteq \cup_{i=1}^m T_i(E)$, and $|T_i(0)| \le n$ for every $i \in \{1, \dots, m\}$. Clearly, $\cup_{m=1}^\infty \cup_{n=1}^\infty sss_{m,n}$ is precisely the set of sub-self-similar sets.

We first prove that $sss_{m,n}$ is closed for every $m, n \in \mathbb{N}$. Suppose that $E_k \to E$ in the Hausdorff metric, where $E_k \in sss_{m,n}$ for every $k \in \mathbb{N}$. To each E_k corresponds $\{T_i^k\}_{i=1}^m$ such that $1/n \le r(T_i^k) \le 1 - 1/n$, $E_k \subseteq \cup_{i=1}^m T_i^k(E_k)$, and $|T_i^k(0)| \le n$. Using the standard matrix, vector representation of an affine transformation, each T_i^k may be associated with a point, x_i^k, in $\mathbb{R}^{d^2+d}$. The conditions on each T_i^k ensure that the set of all such points, K, is compact. By recursively choosing successively finer subsequences, we may assume that each sequence $\{x_i^k\}_{k=1}^{\infty}$ is convergent to say $x_i \in K$. Each point x_i in turn defines a contractive similarity $T_i : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $\frac{1}{n} \leq r(T_i) \leq 1 - \frac{1}{n}$ and $|T_i(0)| \leq n$ for every $i \in \{1, \ldots, m\}$. The correspondence between affine transformations on $\mathbb{R}^d$ and points in $\mathbb{R}^{d^2+d}$, along with the continuity of the algebraic operations, implies that $T_i^k \to T_i$ pointwise as $k \to \infty$. We must now show that $E \subseteq \cup_{i=1}^m T_i(E)$. Let $x \in E$. Then for every $k \in \mathbb{N}$, there is an $x_k \in E_k$ such that the sequence $\{x_k\}_{k=1}^{\infty}$ converges to x. Since $E_k \subseteq \cup_{i=1}^m T_i^k(E_k)$, there is an $i_k \in \{1, \ldots, m\}$ such that $x_k \in T_{i_k}^k(E_k)$. Since there are only finitely many choices for i_k, at least one must occur infinitely often. Thus we have a subsequence $\{k_j\}_{j=1}^{\infty}$ and a fixed $i \in \{1, \ldots, m\}$ such that $i_{k_j} = i$ for every j. Along this subsequence we have

$$T_{i_{k_j}}^{k_j}(E_{k_j}) = T_i^{k_j}(E_{k_j}) \to T_i(E)$$

as $j \to \infty$, since $T_i^{k_j} \to T_i$ pointwise and $E_{k_j} \to E$ in the Hausdorff metric. Thus $x \in T_i(E)$ since $x_{k_j} \to x$ and $x_{k_j} \in T_i^{k_j}(E_{k_j})$ for all j.

Finally, we prove that $sss_{m,n}$ is nowhere dense in $\mathcal{K}(\mathbb{R}^d)$ for all $m, n \in \mathbb{N}$. Since $sss_{m,n}$ is closed, we must simply show that it contains no open set. But this is immediate since its complement is dense in $\mathcal{K}(\mathbb{R}^d)$ by Corollary 2.1.□

The next theorem states a similar result for Sss.

Theorem 3.2 *The set of super-self-similar sets may be expressed as the countable union of closed, nowhere dense subsets of* $\mathcal{K}(\mathbb{R}^d)$.

Proof: The proof of this theorem is very similar to the proof of Theorem 3.1. For $m, n \in \mathbb{N}$, define $Sss_{m,n}$ to be the set of all those super-self-similar sets E such that there exists contractive similarities $\{T_i\}_{i=1}^m$ with $\frac{1}{n} \leq r(T_i) \leq 1 - \frac{1}{n}$, $E \supseteq \cup_{i=1}^m T_i(E)$, and $|T_i(0)| \leq n$ for every $i \in \{1, \ldots, m\}$. Using the exact construction from Theorem 3.1, we obtain a sequence of sets $E_k \to E$ and a sequence of transformations $\{T_i^k\}_{k=1}^{\infty}$, for each $i \in \{1, \ldots, m\}$ satisfying $1/n \leq r(T_i^k) \leq 1 - 1/n$, $E_k \supseteq \cup_{i=1}^m T_i^k(E_k)$, and $|T_i^k(0)| \leq n$. As before, there are transformations $\{T_i\}_{i=1}^m$ which are the pointwise limits as $k \to \infty$ of $\{T_i^k\}_{k=1}^{\infty}$, for each $i \in \{1, \ldots, m\}$ and which satisfy $\frac{1}{n} \leq r(T_i) \leq 1 - \frac{1}{n}$ and $|T_i(0)| \leq n$. We must now show that $E \supseteq \cup_{i=1}^m T_i(E)$. Suppose that $x \in T_i(E)$ for some $i \in \{1, \ldots, m\}$. Since $E_k \to E$ in the Hausdorff metric, $T_i(E_k) \to T_i(E)$ by the continuity of T_i. Thus for each k we may choose $x_k \in E_k$ such that $T_i(x_k) \to T_i(x)$. Thus $x_k \to x$ by the continuity of T_i^{-1} and $x \in E$.

In order to show that $Sss_{m,n}$ is nowhere dense in $\mathcal{K}(\mathbb{R}^d)$ it again suffices to show that it contains no open set, since it is closed. But this follows immediately from Corollary 2.2.□

The above theorems may be somewhat improved. By allowing more general affine contractions, rather than strict similarities, we obtain the notions of sub-self-affinity and super-self-affinity. The above proofs clearly apply to the larger sets of sub-self-affine sets and super-self-affine sets.

We now turn our attention to the set of self-similar sets. It is well known that ss is dense in $\mathcal{K}(\mathbb{R}^d)$. This is essentially the content of the collage theorem (see [Ba] section 3.10, theorem 1). This implies that sss and Sss are dense in $\mathcal{K}(\mathbb{R}^d)$ as they both contain ss. In fact, $ss = sss \cap Sss$. This implies that ss is a first category, F_σ subset of $\mathcal{K}(\mathbb{R}^d)$. Finally, we are interested in the size of ss compared to sss and Sss. As sss and Sss are not G_δ subsets of $\mathcal{K}(\mathbb{R}^d)$, it makes no sense to consider the Baire category of their subsets (see [Ox], chapter 12). Thus we content ourselves with the following theorem which states that ss is a small subset of both sss and Sss.

Theorem 3.3 *$sss \setminus ss$ is dense in sss and $Sss \setminus ss$ is dense in Sss.*

Proof: The first part is quite simple since any finite set is sub-self-similar. The finite sets are dense in $\mathcal{K}(\mathbb{R}^d)$ and, therefore, dense in $sss \setminus ss$.

The second part is slightly more difficult. It suffices to find a class of super-self-similar sets which are not self-similar, but are dense in $\mathcal{K}(\mathbb{R}^d)$. Since ss is dense in $\mathcal{K}(\mathbb{R}^d)$, we show how to approximate any self-similar set with a super-self-similar set which is not self-similar. Let E be self-similar for the transformations $\{T_i\}_{i=1}^m$. Choose $R > 0$ such that $T_i(B_R(0)) \subseteq B_R(0)$ for each $i \in \{1, \ldots, m\}$. Let $E_1 = \cup_{i=1}^m T_i(B_R(0))$ and for $n > 1$ let $E_n = \cup_{i=1}^m T_i(E_{n-1})$. Then each E_n is super-self-similar, but not self-similar and $E_n \to E$ in the Hausdorff metric.$\square$

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