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THE PREVALENT DIMENSION OF GRAPHS

Abstract

We show that the upper entropy dimension of the prevalent function $f \in C[0,1]$ is 2.

1 Prevalence

The extension of the various notions of "almost every" in $\mathbb{R}^n$ to infinite dimensional spaces is an interesting and difficult problem. Perhaps the simplest and most successful generalization has been through the use of category. Banach's application of category to the investigation of differentiability is classic. As another example, [HP] demonstrates that the graph of the generic function has lower entropy dimension one and upper entropy dimension two. There are fundamental difficulties, however, with attempts to extend measures to infinite dimensional spaces. Prevalence is a notion defined in [HSY] which generalizes the measure theoretic "almost every" without actually defining a measure on the entire space. An equivalent notion was originally introduced in [Chr] as pointed out in [HSY2]. Prevalence is defined as follows: Let V be a Banach space. A Borel set $A \subset V$ will be called shy if there is a positive Borel measure μ on V such that $\mu(A+v)=0$ for every $v \in V$. More generally, a subset of a shy Borel set will be called shy. In [HSY] it is shown that shyness satisfies all the properties one would expect of a generalization of measure zero. For example:

1. Shyness is shift invariant.

2. Shyness is closed under countable unions.

3. A subset of a shy set is shy.

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4. A shy set has empty interior.

5. If $V = \mathbf{R}^n$, then the shy sets coincide with the measure zero sets.

The complement of a shy set will be called prevalent. The goal here is to investigate the prevalent dimensional properties of graphs of functions.

2 Dimension

In this section, we define the upper entropy index, Δ, and from that the upper entropy dimension, $\widehat{\Delta}$. For $\varepsilon > 0$, the ε-square mesh for R^2 is defined as the collection of closed squares $\{[i\varepsilon, (i+1)\varepsilon] \times [j\varepsilon, (j+1)\varepsilon]\}_{i,j \in \mathbf{Z}}$. For a totally bounded set $E \subset \mathrm{R}^2$, define

$$N_\varepsilon(E) = \# \text{ of } \varepsilon\text{-mesh squares which meet } E$$

and

$$\Delta(E) = \limsup_{\varepsilon \to 0} \frac{\log N_\varepsilon(E)}{-\log \varepsilon}.$$

An easy but important property of Δ is that it respects closure. That is $\Delta(E) = \Delta(\overline{E})$. Another ([F] p. 41) is that the limsup need only be taken along any sequence $\{c^n\}_{n=1}^\infty$ where $c \in (0,1)$ and we still obtain the same value. One problem with Δ is that it is not σ-stable. In other words it is possible that $\Delta(\cup_n E_n) > \sup_n\{\Delta(E_n)\}$. For example, $\Delta(\mathrm{Q}) = 1$ even though Q is countable. For this reason, Δ is used to define a new set function, $\widehat{\Delta}$, defined by:

$$\widehat{\Delta}(E) = \inf\{\sup_n\{\Delta(E_n)\} : E = \cup_n E_n\}.$$

This new σ-stable set function, $\widehat{\Delta}$, is the upper entropy index. See [Edg] section 6.5 or [F] sections 3.1 through 3.3 for reference.

We may now state the main result. Let $C[0,1]$ denote the Banach space of continuous, real valued functions defined on $[0,1]$ with the uniform metric ρ. For $f \in C[0,1]$, let $G(f) = \{(x, f(x)) : x \in [0,1]\}$ denote the graph of f.

Theorem 2.1 The set $\{f \in C[0,1] : \widehat{\Delta}(G(f)) = 2\}$ is a prevalent subset of $C[0,1]$.

3 Application

In this section, we prove several lemmas and Theorem 2.1. First we fix some notation. Let $I = [k2^{-m}, (k+1)2^{-m}] \subset [0,1]$ be a dyadic interval, where $k, m \in \mathrm{N}$ are fixed. For $f \in C[0,1]$, let $G_I(f) = \{(x, f(x))\}_{x \in I}$ be that portion of the graph of f lying over I. For any interval $[a,b] \subset [0,1]$ define $R_f[a,b] = \sup\{|f(x) - f(y)| : a < x, y < b\}$. For $n > m$, let

$$M_{2^{-n}}(f) = 2^n \sum_{i=k2^{n-m}}^{(k+1)2^{n-m}-1} R_f[i2^{-n}, (i+1)2^{-n}].$$

For $\gamma \in [1,2)$, let $A_\gamma = \{f \in C[0,1] : \Delta(G_I(f)) > \gamma\}$.

Lemma 3.1 For every $f \in C[0,1]$ and natural number $n > m$,

$$M_{2^{-n}}(f) \leq N_{2^{-n}}(G_I(f)) \leq 2^{n-m+1} + M_{2^{-n}}(f).$$

Proof: See [F] proposition 11.1.□

Corollary 3.1 For every $f \in C[0,1]$,

$$\Delta(G_I(f)) = \limsup_{n\to\infty} \frac{\log M_{2^{-n}}(f)}{\log 2^n}.$$

Proof: Note that $\lim_{n\to\infty} 2^{-n} M_{2^{-n}}(f) = \infty$. Thus

$$1 \leq \frac{N_{2^{-n}}(G_I(f))}{M_{2^{-n}}(f)} \leq \frac{2^{n-m-1} + M_{2^{-n}}(f)}{M_{2^{-n}}(f)} \to 1$$

or $M_{2^{-n}} \sim N_{2^{-n}}(G_I(f))$ as $n \to \infty$. The result easily follows.□

Lemma 3.2 The set A_γ is a $G_{\delta\sigma}$ subset of $C[0,1]$.

Proof: For any rational number $q \in (\gamma, 2)$ and any natural number $n > m$, let

$$A_q(n) = \{f \in C[0,1] : \frac{\log M_{2^{-n}}(f)}{\log 2^n} > q\}.$$

Note that

$$A_\gamma = \bigcup_{q\in\mathbb{Q}\cap(\gamma,2)} \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_q(n).$$

So it suffices to show that $A_q(n)$ is an open set.

Let $f \in A_q(n)$. Choose $\varepsilon > 0$ so small that

$$\frac{\log(M_{2^{-n}}(f) - \varepsilon)}{\log 2^n} > q.$$

Suppose that $g \in C[0,1]$ satisfies $\rho(f,g) < \varepsilon 2^{-n}$. Then the triangle inequality yields

$$|f(x) - f(y)| \leq |f(x) - g(x)| + |g(x) - g(y)| + |g(y) - f(y)|.$$

Thus

$$|g(x)-g(y)| \geq |f(x)-f(y)| - 2\varepsilon 2^{-n} \geq |f(x)-f(y)| - \varepsilon 2^{m-n}.$$

Therefore

$$R_g[i2^{-n},(i+1)2^{-n}] \geq R_f[i2^{-n},(i+1)2^{-n}] - 2^{m-n}\varepsilon$$

and

$$\frac{\log(M_{2^{-n}}(g))}{\log 2^n} \geq \frac{\log(M_{2^{-n}}(f)-\varepsilon)}{\log 2^n} > q.$$

Thus $g \in A_q(n)$ and $A_q(n)$ is open.□

Lemma 3.3 For all $f \in C[0,1]$ and $\lambda \neq 0$, $\Delta(G_I(f)) = \Delta(G_I(\lambda f))$.

Proof: This is a simple consequence of the fact that $R_{\lambda f}[a,b] = \lambda R_f[a,b]$.□.

Lemma 3.4 For all $f, g \in C[0,1]$,

$$\Delta(G_I(f+g)) \leq \max\{\Delta(G_I(f)), \Delta(G_I(g))\}.$$

Proof: This is a simple consequence of the inequality

$$R_{f+g}[a,b] \leq R_f[a,b] + R_g[a,b] \leq 2\max\{R_f[a,b], R_g[a,b]\}.□$$

Lemma 3.5 For all $\gamma < 2$, A_γ is a prevalent, Borel set.

Proof: A_γ is a Borel set by lemma 3.2. Let $g \in C[0,1]$ satisfy $\Delta(G_I(g)) > \gamma$. The existence of such a g is guaranteed by the fact that the typical $g \in C[0,1]$ satisfies $\Delta(G_I(g)) = 2$ (see [HP], Proposition 2). Let μ be the Lebesgue type measure concentrated on the line $[g]$ defined by $[g] = \{\lambda g \in C[0,1] : \lambda \in [0,1]\}$. Let $h \in C[0,1]$. We will show that $\#\{(A_\gamma^c + h) \cap [g]\} = 1$. Therefore, $\mu(A_\gamma^c + h) = 0$. Suppose that f_1, $f_2 \in A_\gamma^c$ are such that $f_1 + h \in [g]$ and $f_2 + h \in [g]$. Then there exists $\lambda_1, \lambda_2 \in [0,1]$ such that $f_1 + h = \lambda_1 g$ and $f_2 + h = \lambda_2 g$. This implies $h = \lambda_1 g - f_1 = \lambda_2 g - f_2$. Thus $f_1 - f_2 = (\lambda_1 - \lambda_2)g$. This can only happen if $\lambda_1 = \lambda_2$ by lemmas 3.3 and 3.4. Therefore, $f_1 = f_2$. Since h is arbitrary, this says that A_γ^c is a shy set or A_γ is a prevalent set.□

By expressing $\{f \in C[0,1] : \Delta(G_I(f)) = 2\}$ as a countable intersection

$$\{f \in C[0,1] : \Delta(G_I(f)) = 2\} = \bigcap_{\gamma \in \mathbb{Q} \cap (1,2)} A_\gamma,$$

we obtain the following:

Corollary 3.2 The set $\{f \in C[0,1] : \Delta(G_I(f)) = 2\}$ is a prevalent, Borel subset of $C[0,1]$.

Finally, we prove theorem 2.1.

Proof: Let $\{I_n\}_{n=1}^{\infty}$ be an enumeration of the dyadic intervals and let

$$A_n = \{f \in C[0,1] : \Delta(G_{I_n}(f)) = 2\}.$$

Then A_n is a prevalent, Borel set by corollary 3.2, as is $A = \cap_1^{\infty} A_n$, being the countable intersection of prevalent, Borel sets. If

$$B = \{f \in C[0,1] : \widehat{\Delta}(G(f)) = 2\},$$

then we claim that $A \subset B$. Let $f \in A$ and let $G(f) = \bigcup_1^{\infty} E_n$ be a decomposition. Since Δ respects closure, we may assume that the E_n's are closed. Since $G(f)$ is closed, one of the E_n's must be somewhere dense by the Baire category theorem. Therefore, $E_n \supset G_{I_k}(f)$ for some n, k. Thus, $\Delta(E_n) \geq \Delta(G_{I_k}(f)) = 2$ and $\widehat{\Delta}(G(f)) = 2$. Therefore, B is a prevalent set since it is the superset of a prevalent, Borel set.□

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