INTERSECTIONS OF SELF-SIMILAR SETS

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ABSTRACT. We study sets of the form $T \cap g(T)$, where T is self-similar and g is a bijection. Under certain assumptions, this intersection may be described using a digraph IFS.

1. INTRODUCTION

Let T be a self-similar set and let g be a bijection. We wish to study sets of the form $T \cap g(T)$. Natural examples include the intersection of a set with a translated, rotated or reflected copy of itself. In figure 1 for example, we see two copies of Sierpinski type triangles shaded gray; one is a reflection of the other. The dark portion is an image of the intersection and was generated using a digraph IFS. The goal of this paper is to outline how and when a digraph IFS can be used to generate an such an intersection. We will need to make some fairly strong assumptions to be achieve our objectives. In particular, the IFS generating the given self-similar set together with the function g will need to interact nicely with some lattice. In spite of this, we are still able to generate a fairly rich set of examples.

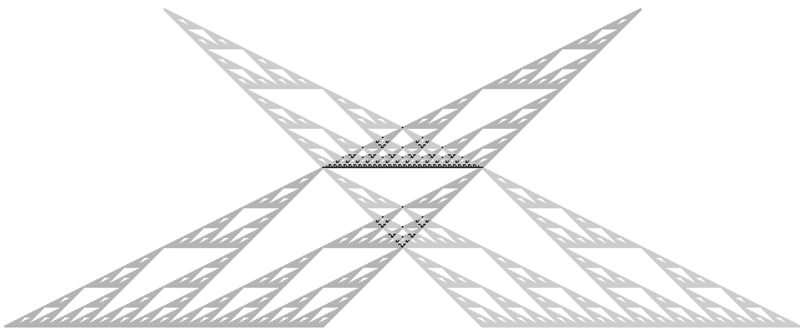

FIGURE 1. The intersection of two Sierpinski triangles

This paper is organized as follows. In section 2, we will define our terms and setting, describe the basic construction, and demonstrate the construction with a few examples. In section 3, we present a natural generalization involving intersections of digraph self-similar sets. The example in this section describes a portion of the boundary of an aperiodic pair of tiles. In fact, the study of self-affine tiles, as in [13], was the starting point for the current paper. Finally in section 4, we discuss separation properties used for computation of fractal dimension.

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2. The setting and basic construction

2.1. **Definitions.** The basic tool for constructing self-similar sets is the *iterated function system* or *IFS*. The basic theory of iterated function systems was laid out in [7] and [5, 6] are standard references. We will work exclusively in Euclidean space $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called a *contraction* if there is a number $r \in (0,1)$ so that $|f(x)-f(y)| \leq r|x-y|$. We will deal exclusively with bijective contractions. An iterated function system or IFS on $\mathbb{R}^n$ is a non-empty set of contractions of $\mathbb{R}^n$. A now classic result in fractal geometry states that given an IFS $\{f_i\}_{i=1}^m$, there is always a unique non-empty, compact set $T \subset \mathbb{R}^n$ so that

$$T = \bigcup_{i=1}^m f_i(T).$$

The set T is called the *invariant set* of the IFS. If each f_i is a similarity (i.e. $|f_i(x) - f_i(y)| = r_i|x-y|$, for all x, y), then T is called *self-similar*. In this case, the list of numbers $\{r_i\}_{i=1}^m$ is called the *similarity ratio list* of the IFS. If each f_i is an affine function, then T is called a *self-affine set*.

In order to compute the dimension of a self-similar set, some sort of separation condition should be satisfied. Of course, by dimension we mean the Hausdorff dimension, which is equal to the box-counting dimension for self-similar sets. The simplest and most well known separation condition is the *open set condition* or *OSC*. An IFS satisfies OSC if there is a non-empty open set U such that

$$U \supset \bigcup_{i=1}^m f_i(U),$$

with this union disjoint. A fundamental result in fractal geometry states that if T is a self-similar set generated by an IFS satisfying OSC and with ratio list $\{r_i\}_{i=1}^m$, then the dimension of T is the unique number $s > 0$ so that

$$\sum_{i=1}^m r_i^s = 1.$$

When there is a single number $r > 0$ so that $r_i = r$ for each i, then this formula states that the dimension is $-\log(m)/\log(r)$.

Now suppose that T is the invariant set of an IFS consisting of bijective contractions and that $g : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection. In order to describe $T \cap g(T)$, we need to generalize the notion of IFS to that of a *digraph IFS*. The notion of a digraph iterated function system was formulated in [11] and an exposition appears in [5]. The adjectives mixed [1] and generalized [13] have also been used. A *Mathematica* package to generate sets using digraph iterated function systems is described in [12].

The first component to define a digraph IFS is a directed multi-graph G that consists of a finite set V of vertices and a finite set E of directed edges between vertices. Given two vertices u and v, denote the set of all edges from u to v by E_{uv}. Given $e \in E_{uv}$, we say that u is the *initial vertex* of e and v is the *terminal vertex* of e. A *path* through G is a finite sequence of edges so that the terminal vertex of any edge is the initial vertex of the subsequent edge. The set of paths from u to v is denoted by $E_{uv}^{(*)}$. A *loop* is any path which begins and ends at the same vertex. G is called strongly connected if for every $u, v \in E$, there is a path from u to v.

We now form a digraph IFS by associating a metric space X_u with each vertex $u \in V$. In this paper, $X_u = \mathbb{R}^n$ for each u, but they should still be thought of as distinct spaces. Also, for each $u, v \in V$ and $e \in E_{uv}$, we associate a function $f_e : X_v \to X_u$. Given a path α through G, we may form a function f_α by composing the functions f_e over $e \in \alpha$ taken in reverse order along the path α. If $\alpha \in E_{uv}^{(*)}$, then $f_\alpha : X_v \to X_u$. If each f_e is a similarity with similarity ratio r_e, then the similarity ratio of f_α is simply the product of the r_e over $e \in \alpha$ and is denoted r_α. The digraph IFS is called *contractive* if $r_\alpha < 1$ for every loop α. Given a contractive digraph IFS, there is a unique set of compact sets T_v, one for every $v \in V$, such that for every $u \in V$

$$T_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} f_e(T_v). \tag{1}$$

Such a collection of sets is called the *invariant list* of the digraph IFS. If each function f_e is a similarity, then the members of the invariant list are called *digraph self-similar sets.*

As with self-similar sets, the dimension of digraph self-similar sets may be computed provided some separation condition is satisfied. A digraph IFS is said to satisfy the open set condition if there exist open sets U_v, one for every $v \in V$, such that for every $u \in V$

$$U_u \supset \bigcup_{v \in V} \bigcup_{e \in E_{uv}} f_e(U_v),$$

with this union disjoint. When a digraph IFS arising from a strongly connected digraph satisfies OSC and each f_e is a similarity with common similarity ratio r, then the dimension of all sets in the invariant list may be computed using the *substitution matrix* of the digraph IFS. The substitution matrix M is simply the adjacency matrix representation of the digraph, i.e. $M_{uv} = \#E_{uv}$. In this case, the common dimension of the digraph self-similar sets is $-\log(\lambda)/\log(r)$, where λ is the spectral radius of M.

Note that a digraph IFS itself may be thought of as a type of matrix, which we denote M^*. The rows and columns of M^* are indexed by the vertices V. The entry M^*_{uv} in row u and column v should be the set of functions mapping $X_v \to X_u$. Using this notation, we may write equation 1 alternatively as

$$T_u = \bigcup_{v \in V} \bigcup_{f \in M^*_{uv}} f(T_v). \tag{2}$$

This is essentially the representation used to store a digraph IFS in [12]. It is also particularly convenient when V is a large set, as will often happen in our construction.

2.2. **The main construction.** The technique to find a digraph IFS to generate an intersection of the form $T \cap g(T)$ is described in the proof the following theorem. This theorem is essentially a direct generalization of the technique used to describe the boundary of a tile in [13].

Theorem 1. *Let T be the invariant set of an IFS $\{f_i\}_{i=1}^m$ consisting of bijective contractions. Suppose that S is a finite set of bijections so that for all $g \in S$ satisfying $T \cap g(T) \neq \varnothing$ and for all $i, j = 1, \ldots, m$ satisfying $T \cap f_i^{-1} g f_j(T) \neq \varnothing$ we have $f_i^{-1} g f_j \in S$. Then the collection of sets $\{T \cap g(T) : g \in S\}$ forms the invariant list of a digraph IFS.*

Proof. We simply perform a few basic set theoretical operations using the fact that all functions involved are bijections. Given any $g \in S$,

$$\begin{aligned} T \cap g(T) &= \left[\bigcup_{i=1}^{m} f_i(T)\right] \cap \left[g\left(\bigcup_{j=1}^{m} f_j(T)\right)\right] \\ &= \bigcup_{i,j=1}^{m} \left(f_i(T) \cap g f_j(T)\right) \\ &= \bigcup_{i,j=1}^{m} f_i\left(T \cap f_i^{-1} g f_j(T)\right). \end{aligned}$$

First note that if $T \cap g(T) = \varnothing$, then the above calculation shows that $T \cap f_i^{-1} g f_j(T) = \varnothing$ for each i, j. Even if $T \cap g(T) \neq \varnothing$, many of these sets $T \cap f_i^{-1} g f_j(T)$ may be empty. For those that are not empty however, the corresponding function $f_i^{-1} g f_j$ is in S. Given $g, h \in S$, let $C(g,h) = \{(i,j) : h = f_i^{-1} g f_j\}$. Then

$$T \cap g(T) = \bigcup_{h \in S} \bigcup_{(i,j) \in C(g,h)} f_i(T \cap h(T)). \tag{3}$$

Now equation 3 has the same form as equation 1. The vertices of the digraph may be taken to be the elements of S. The directed edges from g to h correspond to the elements of $C(g,h)$. If $(f_i, f_j) \in C(g,h)$, then the function f_i is associated with that directed edge. The digraph IFS is clearly contractive since each f_i is a contraction. Finally, equation 3 states that $\{T \cap g(T) : g \in S\}$ forms the invariant list of this digraph IFS. □

In practice, we start with a single function g and are interested in $T \cap g(T)$. Thus we need to find a set of functions S containing g and satisfying the hypotheses of the theorem. This will typically be done using an iterative procedure. Start with with $S_0 = \{g\}$ and recursively define S_{k+1} using a two step process. First, let

$$S_{k+1}^{(1)} = S_k \cup \{f_i^{-1} h f_j : h \in S_k \text{ and } i, j = 1, \ldots, m\}.$$

Then form S_{k+1} by selecting those elements h from $S_{k+1}^{(1)}$ such that $T \cap h(T) \neq \varnothing$. Iterate the procedure until no new functions are produced. If T is a complicated set, then it may be difficult to determine if $T \cap h(T) \neq \varnothing$. To simplify the process, we may use some super-set $T_0 \supset T$ in place of T. For example, T_0 may be the convex hull of T; determination of the intersection of convex sets is a relatively easy problem. This may result in a set S which is too large, i.e. there may be functions $h \in S$ such that $T \cap h(T) = \varnothing$. Note however that theorem 1 allows for this possibility. When the digraph IFS describing the intersection is constructed, vertices with no edges leaving them will correspond to empty sets. Vertices with edges going only to empty vertices will also be empty.

As a first example, we intersect the Cantor set with a scaled and translated copy of itself. Several authors have studied the Cantor set intersected with a translate of itself [4, 9, 10]. Our example is a variation of this problem. Let C denote the Cantor set. Specifically, C is the invariant set of the IFS containing the two real functions $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$. We will determine a digraph IFS to generate the set $C \cap (\frac{1}{4}C + \frac{1}{2})$. In the language of theorem 1, we take $g(x) = x/4 + 1/2$. Note that this set is non-empty as it contains $3/4$, although the algorithm will yield this fact independently. Now if we write $f_i(x) = x/3 + b_i$ and similarly for f_j, we have

$$f_i^{-1} g f_j(x) = \frac{1}{4}x + 3\left(\frac{1}{4}b_j + \frac{1}{2} - b_i\right).$$

In order for this function to be in our set S, we need $3(\frac{1}{4}b_j + \frac{1}{2} - b_i) \in [\text{-}1/4, 1]$. To form the set S_1, there are four possible combinations of b_i and b_j to consider.

$$\begin{aligned}
&1)\ b_i = 0,\ b_j = 0 &&\Longrightarrow\ 3\left(\frac{1}{4}\,0 + \frac{1}{2} - 0\right) = \frac{3}{2} \notin [\text{-}\frac{1}{4}, 1]\\
&2)\ b_i = \frac{2}{3},\ b_j = 0 &&\Longrightarrow\ 3\left(\frac{1}{4}\,0 + \frac{1}{2} - \frac{2}{3}\right) = \text{-}\frac{1}{2} \notin [\text{-}\frac{1}{4}, 1]\\
&3)\ b_i = 0,\ b_j = \frac{2}{3} &&\Longrightarrow\ 3\left(\frac{1}{4}\,\frac{2}{3} + \frac{1}{2} - 0\right) = 2 \notin [\text{-}\frac{1}{4}, 1]\\
&4)\ b_i = \frac{2}{3},\ b_j = \frac{2}{3} &&\Longrightarrow\ 3\left(\frac{1}{4}\,\frac{2}{3} + \frac{1}{2} - \frac{2}{3}\right) = 0 \in [\text{-}\frac{1}{4}, 1]
\end{aligned}$$

Thus $S_1 = \{g, h\}$, where $h(x) = x/4$. Applying this same process to h, we find

$$\begin{aligned}
&1')\ b_i = 0,\ b_j = 0 &&\Longrightarrow\ 3\left(\frac{1}{4}\,0 + 0 - 0\right) = 0 \in [\text{-}\frac{1}{4}, 1]\\
&2')\ b_i = \frac{2}{3},\ b_j = 0 &&\Longrightarrow\ 3\left(\frac{1}{4}\,0 + 0 - \frac{2}{3}\right) = \text{-}2 \notin [\text{-}\frac{1}{4}, 1]\\
&3')\ b_i = 0,\ b_j = \frac{2}{3} &&\Longrightarrow\ 3\left(\frac{1}{4}\,\frac{2}{3} + 0 - 0\right) = \frac{1}{2} \in [\text{-}\frac{1}{4}, 1]\\
&4')\ b_i = \frac{2}{3},\ b_j = \frac{2}{3} &&\Longrightarrow\ 3\left(\frac{1}{4}\,\frac{2}{3} + 0 - \frac{2}{3}\right) = \text{-}\frac{3}{2} \notin [\text{-}\frac{1}{4}, 1]
\end{aligned}$$

We have not generated any new functions and the process terminates. In this case, $S = S_1 = \{g, h\}$.

We can now use this information to construct the digraph IFS M^*. Given any pair (g_1, g_2) chosen from S, find all pairs (f_i, f_j) so that $g_2 = f_i^{-1} g_1 f_j$. Any such occurrence generates a directed edge from g_1 to g_2 labeled by the function f_i in the digraph IFS. In the matrix representation, this means that $f_i \in M^*_{g_1 g_2}$. For example, $M^*_{hg} = \{f_1\}$ since $g = f_i^{-1} h f_j$ precisely when $i = 1$ and $j = 2$. This observation corresponds to case (3') above. We may write down the digraph IFS defining both intersections $C \cap g(C)$ and $C \cap h(C)$ in matrix form as follows. First suppose that the rows and columns are indexed by S in the order (h, g). Then the full digraph IFS is

$$M^* = \begin{pmatrix} \{f_1\} & \{f_1\} \\ \{f_2\} & \varnothing \end{pmatrix}.$$

It is easy to see that the digraph open set condition is satisfied using the open intervals $(0, 1/4)$ and $(1/2, 3/4)$. Thus we may use the substitution matrix M to compute the common dimension of these sets. Note that the entry $M_{g_1 g_2}$ simply counts the number of elements in $M^*_{g_1 g_2}$. Thus

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The spectral radius of M is the golden ratio φ and the dimension of the sets is $\log \varphi / \log 3$.

A few comments concerning this construction are in order.

(1) Let $g(x) = mx + b$ where m and b are rational. If Γ is a one-dimensional lattice containing b, $2/3$, and $2m/3$, then $3\,(mb_j + b - b_i)$ will be in Γ. In fact, any subsequent shifts will also be in Γ and the process will terminate since there are only finitely many shifts $b' \in \Gamma$ so that $C \cap (mC + b') \neq \varnothing$.

(2) On the other hand, if either m or b are irrational, then the iteration will not terminate and this technique will not work.

(3) The set $C \cap g(C)$ is actually a self-similar set. In fact, it is the invariant set of the IFS $\{f_2 f_1, f_2 f_1 f_1\}$. This may be deduced from the digraph IFS above by composing the functions along two paths from node g to itself. In general, the digraph IFSs we generate may frequently be simplified, but not typically to the point of a single self-similar set.

2.3. **Intersections in the plane.** Suppose that T is a self-affine set in the plane and $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a non-singular affine function. Theorem 1 supplies a digraph IFS to generate $T \cap g(T)$, provided the iterative procedure to find S terminates. We wish to outline conditions sufficient to guarantee the existence of S. If we write $f_j(x) = A_j x + b_j$, $f_i^{-1}(x) = A_i^{-1}(x - b_i)$, and $g(x) = Ax + b$, then we find that

$$f_i^{-1} g f_j(x) = A_i^{-1} A A_j x + A_i^{-1}(Ab_j + b - b_j). \tag{4}$$

Using the general form of equation 4, it is fairly easy to devise conditions conditions to ensure existence of S. In the following lemma, $\mathcal{A} = \{A_i\}_{i=1}^m$ and we act on $\mathcal{A}$ as a set. For example, $\mathcal{A}^{-1} = \{A_i^{-1}\}_{i=1}^m$.

Lemma 2. *Suppose that there is a lattice $\Gamma \subset \mathbb{R}^2$ which contains b, b_i, and $A\,b_i$ for all i and satisfies $\mathcal{A}^{-1}\Gamma \subset \Gamma$. Suppose further that there is a discrete set $\Delta \subset GL_2(\mathbb{R})$ containing A and satisfying $\mathcal{A}^{-1}\Delta\mathcal{A} \subset \Delta$. Then the iteration procedure to find S will terminate and $T \cap g(T)$ will be a digraph self-affine set.*

The proof is a simple consequence of the facts that all functions generated are elements of a discrete set and the convex hull of a self-similar set is bounded.

The overall process is conceptually simple, but tedious; it may be implemented on a computer. To illustrate the main ideas we discuss the digraph IFS used to generate figure 1. The initial set T is a Sierpinski triangle with vertices at the points $(0,0)$, $(-2,0)$, and $(2,2)$. Specifically, T is generated by the IFS consisting of the following three functions.

$$\begin{aligned} f_1(x) &= x/2 \\ f_2(x) &= x/2 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\ f_3(x) &= x/2 + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned}$$

The set T is intersected with a reflection of itself. The function g is defined by

$$g(x) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The hypotheses of lemma 2 are satisfied by choosing $\Gamma = \mathbb{Z}^2$ and $\Delta = \{A\}$. Using the iterative procedure described earlier, we find that S consists of all functions of

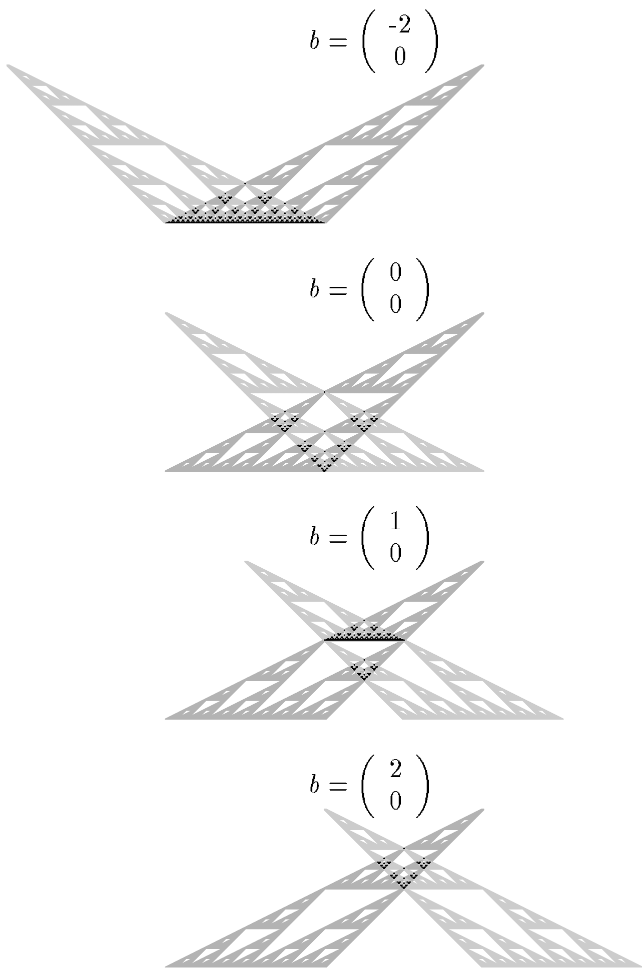

FIGURE 2. The four interesting intersections to generate figure 1

the form $h(x) = Ax + b$ where b is one of.

$$\begin{pmatrix} \text{-}2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} \text{-}4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \text{-}2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ \text{-}2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \text{ or } \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

With this information, the digraph IFS can be set up. The rows and columns will be indexed by the list of functions above. Given a pair of functions (g_1, g_2), we find all pairs of functions (f_i, f_j) satisfying $g_2 = f_i^{-1} g_1 f_j$. If (f_i, f_j) is such a pair, then

$f_i \in M^*_{g_1 g_2}$. In this case we find that the matrix M^* expressing the digraph IFS is

$$\begin{pmatrix}
\{f_1,f_2\} & \{f_2\} & \varnothing & \varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \{f_1\} & \varnothing & \{f_1,f_2\} & \{f_3\} & \{f_3\} & \{f_2\} & \varnothing & \varnothing & \{f_2\} \\
\{f_3\} & \varnothing & \varnothing & \{f_1\} & \varnothing & \{f_3\} & \{f_1\} & \{f_3\} & \{f_2\} & \{f_1,f_2\} \\
\varnothing & \{f_3\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\} & \{f_1\} & \{f_1\} \\
\varnothing & \varnothing & \varnothing & \varnothing & \{f_2\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_2\} & \varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\} & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_1\} & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\} & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\}
\end{pmatrix}$$

The set we are interested in $T \cap g(T)$ is the third of ten digraph self-similar sets. Note however that the last six of these are just single points, because the last six rows of M^* contain just one contraction. These arise because $h(T)$ intersects T at just a vertex. The four interesting intersections are shown in figure 2.

This digraph IFS is not strongly connected, but may still be used to determine the dimension of the sets. The singletons have dimension zero, of course. We may discard the last six rows and columns of M^*, without changing the dimension; the presence of these rows and columns contributes countably many isolated points to the sets in figure 2. The four remaining sets (minus the isolated points) are the invariant sets of the digraph IFS generated by the first four rows and columns of M^*. This smaller digraph IFS is still not strongly connected, but it does satisfy OSC. We may choose the open sets to be $T_0 \cap h(T_0)$, where T_0 is the interior of the triangle with vertices $(0,0)$, $(-2,0)$, and $(2,2)$ and h ranges over the first four functions of S. The second set is simply a one-dimensional self-similar set with respect to the IFS $\{f_1, f_1 f_3, f_2 f_3\}$. The first set is a one-dimensional self-similar set with respect to the IFS $\{f_1, f_2\}$, together with countably many copies of the second set; it also has dimension one. The fourth set is simply a scaled version of the second set and the third consists of a copy of the first and a copy of the fourth. Thus all sets have dimension one.

3. Intersection of digraph self-similar sets

Since we have had to introduce digraph iterated function systems to investigate intersections even when starting with with a simple IFS, it is natural to ask what happens if we start with digraph self-similar sets. Suppose that $\{T_u\}_{u\in V}$ is the invariant list of a digraph IFS satisfying equation 2 (the matrix representation) where each f is a bijective contraction. Recall that we have associated a metric space X_u with each $u \in V$. Now suppose that $g : X_v \to X_u$ is a bijection. The basic technique of this paper is applicable to $T_u \cap g(T_v)$. Indeed, mimicking the proof of theorem 1, we find

$$\begin{aligned}
T_u \cap g(T_v) &= \left[\bigcup_{u'} \bigcup_{f\in M^*_{uu'}} f(T_{u'})\right] \cap \left[g\left(\bigcup_{v'} \bigcup_{h\in M^*_{vv'}} h(T_{v'})\right)\right] \\
&= \bigcup_{u',v'} \bigcup_{f\in M^*_{uu'}} \bigcup_{h\in M^*_{vv'}} f(T_{u'} \cap f^{-1} g h(T_{v'})).
\end{aligned}$$

In this situation, the set of functions S should be replaced by a matrix of functions S indexed by V. S should have the property that whenever $g_1 \in S_{u_1 v_1}$, $f \in M^*_{u_1 u_2}$, $h \in M^*_{v_1 v_2}$, and $g_2 = f^{-1} g_1 h$ satisfies $T_{u_2} \cap g_2(T_{v_2}) \neq \varnothing$, we have $g_2 \in S_{u_2 v_2}$. Given $g_1 \in S_{u_1 v_1}$ and $g_2 \in S_{u_2 v_2}$, let

$$C(g_1, g_2) = \{(f, h) \in M^*_{u_1 u_2} \times M^*_{v_1 v_2} : g_2 = f^{-1} g_1 h\}.$$

We may then write

$$T_{u_1} \cap g_1(T_{v_1}) = \bigcup_{\substack{u_2, v_2 \\ g_2 \in S_{u_2 v_2}}} \bigcup_{(f,h) \in C(g_1, g_2)} f(T_{u_2} \cap g_2(T_{v_2})). \tag{5}$$

Equation 5 again has the same form as equation 1. Let us write $\widehat{M}^*$ for the intersection digraph IFS to distinguish it from the original digraph IFS M^*. Thus we may rewrite equation 5 as

$$T_{u_1} \cap g_1(T_{v_1}) = \bigcup_{\substack{u_2, v_2 \\ g_2 \in S_{u_2 v_2}}} \bigcup_{f \in \widehat{M}^*_{g_1 g_2}} f(T_{u_2} \cap g_2(T_{v_2})).$$

The vertices for $\widehat{M}^*$ correspond to the elements of S. The directed edges from g_1 to g_2 correspond to the elements of $C(g_1, g_2)$. If $(f, h) \in C(g_1, g_2)$, then the function f is associated with that directed edge. Note that if $f \in \widehat{M}^*_{g_1 g_2}$ where $g_1 \in S_{u_1 v_1}$ and $g_2 \in S_{u_2 v_2}$, then $f \in M^*_{u_1 u_2}$ (since $f = g_1 h g_2^{-1}$). It follows that any loop in the digraph for $\widehat{M}^*$ beginning and ending at $g_1 \in M^*_{u_1 v_1}$ must correspond to a loop in the digraph for M^* beginning and ending at u_1. Thus $\widehat{M}^*$ is contractive whenever M^* is contractive.

We illustrate this technique using a pair of tiles with fractal boundary. Figure 3 illustrates the sets T_1 and T_2 under consideration. T_1 and T_2 are the top two figures in figure 3. The bottom two figures show how T_1 and T_2 fit together to form the invariant list of a digraph IFS. These sets were initially described in [2], where it is shown that they form an aperiodic pair of tiles.

More specifically, let us say that T_1 lives in the space X_1 and T_2 lives in the space X_2, where X_1 and X_2 are distinct copies of $\mathbb{R}^2$. Let φ denote the golden ratio and let $R(\theta)$ denote the rotation matrix through the angle θ. T_1 and T_2 are the invariant sets of the digraph IFS M^* defined by

$$M^* = \begin{pmatrix} \{f_1, f_2\} & \{f_3\} \\ \{f_1\} & \{f_3\} \end{pmatrix},$$

where

$$\begin{aligned} f_1(x) &= \frac{1}{\varphi} R(7\pi/5)\, x + \begin{pmatrix} 0 \\ 1/\varphi \end{pmatrix}, \\ f_2(x) &= \frac{1}{\varphi} R(3\pi/5)\, x + \begin{pmatrix} 0 \\ 1/\varphi \end{pmatrix}, \text{ and} \\ f_3(x) &= \frac{1}{\varphi} R(4\pi/5)\, x + \begin{pmatrix} \sin(4\pi/5) \\ (1 - \cos(4\pi/5))/\varphi \end{pmatrix}. \end{aligned}$$

Now suppose that $g : X_2 \to X_1$ is the function

$$g(x) = R(-2\pi/5)x + \begin{pmatrix} \sqrt{2+\varphi}/(2\varphi^2) \\ (2+\varphi)/(2\varphi) \end{pmatrix}.$$

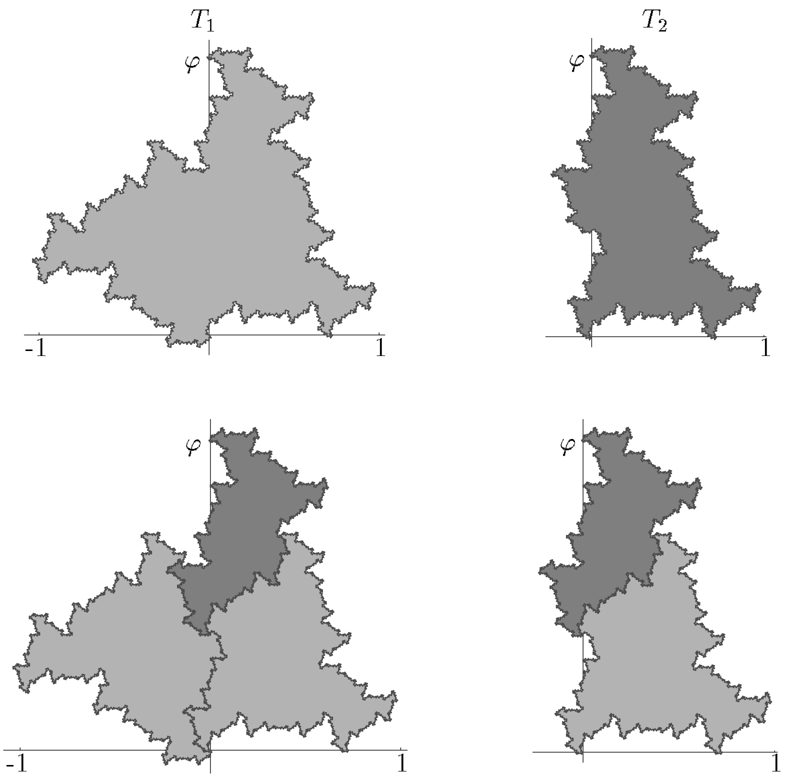

FIGURE 3. A pair of tiles with fractal boundary

We wish to investigate $T_1 \cap g(T_2)$. The sets T_1 and $g(T_2)$ are shown in figure 4. The intersection of these sets forms a portion of the boundary between the two and is highlighted in the picture. We may again find the matrix S using an iterative procedure. In this particular case, we set

$$S_0 = \begin{pmatrix} \varnothing & \{g\} \\ \varnothing & \varnothing \end{pmatrix}.$$

To find S_{k+1}, we again use a two step process. The element in row u_2 and column v_2 of $S_{k+1}^{(1)}$ is

$$S_{k+1,u_2v_2}^{(1)} = (S_{k,u_2v_2}) \cup \left(\bigcup_{u_1,v_1} \bigcup_{\substack{f \in M_{u_1u_2}^* \\ h \in M_{v_1v_2}^*}} \bigcup_{g_1 \in S_{k,u_1v_1}} f^{-1} g_1 h \right).$$

We then form S_{k+1} by choosing from each $S_{k+1,u_2v_2}^{(1)}$ those functions g_2 so that $T_{u_2} \cap g_2(T_{v_2}) \neq \varnothing$. As before, we may choose super sets if desired to simplify the process.

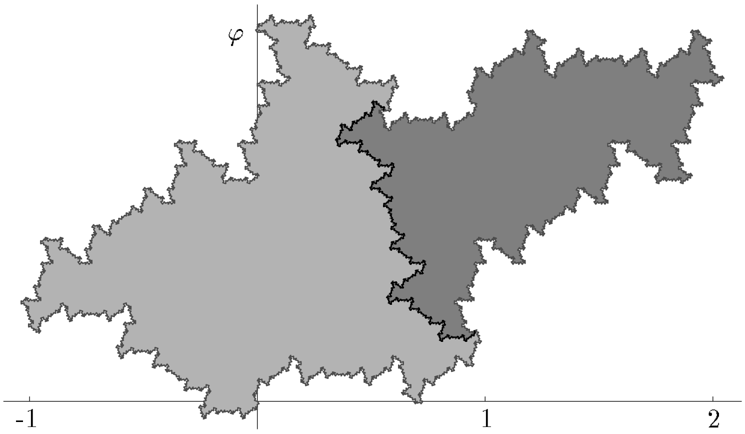

FIGURE 4. The intersection of T_1 with $g(T_2)$

Performing this process in this particular case, we find that S contains a total of 25 functions. Fifteen of these result in single point intersections that are contained in the other ten. The other nine of the ten important intersections are shown in figure 5. Thus the intersection of interest is one of ten digraph self-similar sets; in fact, it is the fifth element of the invariant list of the digraph IFS represented by the following matrix $\widehat{M}^*$:

$$\begin{pmatrix}
\varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\} & \{f_3\} & \varnothing \\
\{f_2\} & \varnothing & \varnothing & \varnothing & \varnothing & \{f_2\} & \varnothing & \varnothing & \varnothing & \{f_3\} \\
\varnothing & \varnothing & \varnothing & \varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\} & \varnothing & \varnothing & \varnothing \\
\varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\} & \varnothing & \varnothing \\
\varnothing & \varnothing & \varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\
\{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing \\
\{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing \\
\varnothing & \{f_1\} & \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & \{f_3\} & \varnothing & \varnothing
\end{pmatrix}.$$

The functions f_1, f_2, and f_3 are the same functions defining the original digraph IFS M^*. A glance at figure 5 reveals that many of these sets are the same up to isometry. In fact, all of the intersections are geometrically similar to one another. It is not difficult to use this fact to express the object of interest as a single self-similar set. Note however that this simplification may only be achieved at the expense of introducing new functions into the digraph IFS. Furthermore, the algorithm we have used may be automated.

Note that $\widehat{M}^*$ is strongly connected and the open set condition is satisfied, as we will show in the next section. Thus we may use the substitution matrix $\widehat{M}$ to

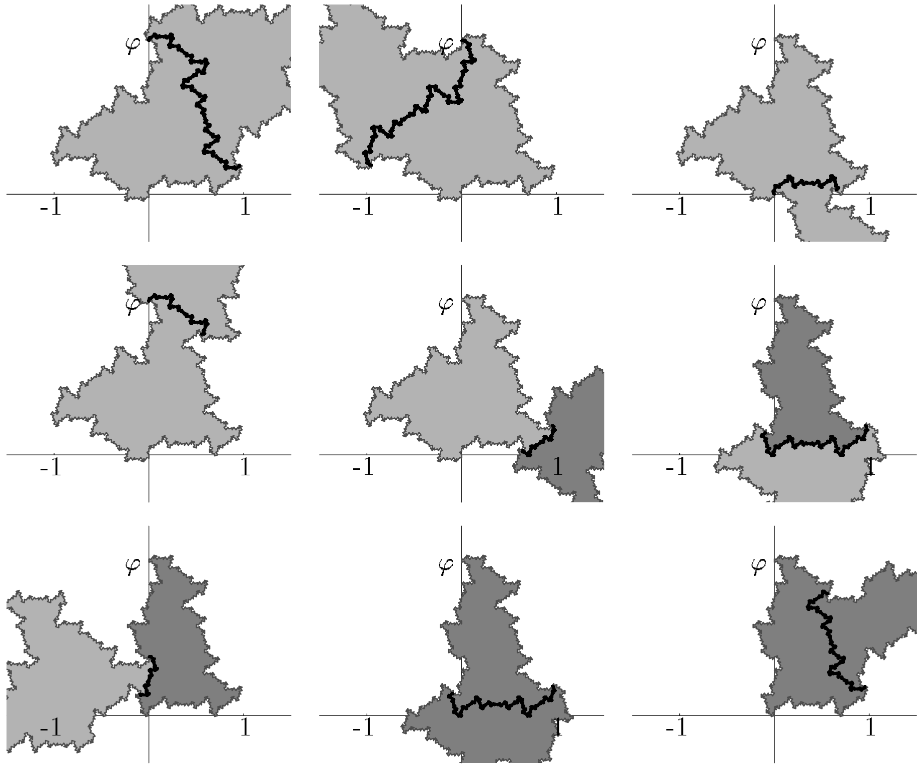

FIGURE 5. The other nine important intersections

compute the dimension of the sets where

$$\widehat{M} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The spectral radius $\lambda \approx 1.839$ of this matrix satisfies $\lambda^3 - \lambda^2 - \lambda - 1 = 0$. The dimension of the set in question is $\frac{\log \lambda}{\log \varphi} \approx 1.266$.

4. SEPARATION PROPERTIES

In order to compute the fractal dimension of a digraph self-similar set, some separation property should be satisfied. In this section, we discuss the open set condition and the weak separation property for digraph iterated function systems generated by the techniques described in this paper. Since a self-similar set may be considered as a digraph self-similar set using a digraph with a single node, we consider only the more general setting described in section 3.

As in section 3, suppose that $\{T_u\}_{u \in V}$ is the invariant list of a digraph IFS M^* and that $g : X_v \to X_u$ is a bijection. Furthermore, suppose the digraph IFS satisfies some separation property. Ideally, we would like the intersection digraph IFS associated with $T_u \cap g(T_v)$ to inherit the separation property. As the examples in this paper show, the situation is not quite so simple for the open set condition. In particular, it is possible that $T_u \subset f(T_v)$ for some $f \in E_{uv}$. This makes it impossible for the open set condition to be satisfied.

Fortunately, there is another separation property, called *the weak separation property* or *WSP*, which is ideally suited for our situation. Furthermore, there is a natural relationship between WSP and OSC which allows us to show that OSC is satisfied under appropriate conditions. The weak separation property in the context of iterated function systems was introduced by Lau and Ngai [8] and studied further by Zerner [14]. The generalization to digraph iterated function systems was carried out by Das and Edgar [3].

The are several equivalent formulations of WSP. We choose a particularly simple formulation here, which is easily applicable to our situation. Although we've primarily used the matrix formulation of a digraph IFS to this point, we now switch to the graph theoretic notation following [3]. Using this notation, as described in section 2, for $u, v \in V$ define

$$F_{uv} = \{f_\alpha^{-1} f_\beta : \alpha, \beta \in E_{uv}^{(*)}\}.$$

The digraph IFS satisfies WSP if the identity is an isolated point of F_{uu} for every $u \in V$.

Now suppose that a digraph IFS G and bijection $g : X_v \to X_u$ generate an intersection digraph IFS $\widehat{G}$. As observed in the previous section, any loop in the digraph for $\widehat{G}$ beginning and ending at f_e for $e \in M_{uv}^*$ must correspond to a loop in the digraph G beginning and ending at u. It follows immediately that $\widehat{G}$ satisfies WSP whenever G satisfies WSP.

We need two more definitions to state the relationship between WSP and OSC. A set $K \subset \mathbb{R}^n$ is said to be in *general position* if it is not contained in any affine subspace of $\mathbb{R}^n$. A digraph IFS G is said to *distinguish paths* if for every $u, v \in V$ and distinct $\alpha, \beta \in E_{uv}^*$, we have $f_\alpha \neq f_\beta$. Note that if G and $\widehat{G}$ are as above and G distinguishes paths, then $\widehat{G}$ distinguishes paths using almost the exact same argument.

The relationship between WSP and OSC is summarized in the following lemma ([3], Proposition 3.1).

Lemma 3. *Let G be strongly connected and suppose the elements of invariant list of G are in general position. Then G satisfies OSC if and only if G has WSP and G distinguishes paths.*

This lemma is immediately applicable to the situation described in the paper. In particular, the reduced digraph IFS generating the intersections described in the previous section does satisfy OSC. The larger digraph IFS which also generates several single point intersections (which are not in general position) does not satisfy OSC.

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