Images of a vibrating Koch drum

Mark McClure

Department of Mathematics
UNC - Asheville
Asheville, NC 28804

Abstract

We illustrate the vibrational modes of a drum shaped like the Koch snowflake. Vibrational modes are computed using a fairly standard finite difference technique applied to a recently published discrete approximation to the snowflake. The computations are all performed in Mathematica 6.0 that has substantial graphical improvements over Mathematica 5.2. In particular, vertex normals may be specified to achieve a smoothing effect of the surface of a 3D vibrational plot and colors may be interpolated throughout polygons to allow a smooth color gradation in contour plots.

Key words: fractal, Laplacian, vibration
PACS:

1 Introduction

The Koch snowflake is a well known self-similar set whose boundary is a fractal curve. Figure 1 illustrates the decomposition of this set into seven copies of itself - six scaled by the factor $1/3$ and one scaled by the factor $1/\sqrt{3}$.

Suppose a vibrating membrane with fixed boundary has the shape of a Koch snowflake. What types of vibrational modes are possible? This natural question was suggested by a conjecture of Berry in 1979 [1] and a number of papers have studied the question since. In the mathematical formulation of the problem, the natural modes of vibrations are described by the eigenfunctions of the Laplacian on the region. Lapidus, Neuberger, Renka, and Griffith [2] approximated the Koch drum with a discrete grid of points to translate the problem into a matrix eigenvalue problem. An improved grid was introduced recently

Email address: mcmcclur@unca.edu (Mark McClure).

in [3] and we use this new grid to along the improved graphical capabilities of Mathematica 6.0 to present some very nice images of the vibrational modes of the Koch drum.

2 Mathematical statement of the problem

The basic wave equation is $w_{tt} = c^2 \triangle w$, where c is a parameter depending upon the material and $w(x,t)$ is a function of time t and of the spatial variable x restricted to lie in some specified domain. Assuming that $w(x,t) = u(x)g(t)$ and separating variables, we obtain the equations $g'' = -\lambda c^2 g$ and $\triangle u = -\lambda u$. Any solution of the equation for u is simply an eigenfunction of the Laplacian on the domain in question and any such u defines a possible shape of the vibration called a fundamental mode. The possible values of λ dictate the frequencies of the fundamental modes.

In the context here, x is a two-dimensional variable permitted to range throughout the Koch snowflake and we assume that $w(x,t) = 0$ for all x on the boundary of the snowflake. There is no simple analytic formula for the solution so we use a numerical technique using finite differences based on the approximating grids shown in Figure 2 and Figure 3. These are the first two levels in a family of approximating grids. The purpose of the numbering and shading will be explained shortly, but there is one potential point of confusion we should clarify right away. The geometry of the grid dictates the location of the exterior vertices and some of these lie on top of one another. For example, there are three vertices right on top of one another at the white vertex in Figure 2; one is adjacent to vertex 7, another is adjacent to vertex 8, and the last is adjacent to vertex 9. This is will be important when we use the exterior points to enforce the boundary condition.

Fix some level of the approximation and let x be an interior point of this grid - i.e. an element of the grid that is inside the snowflake. We will refer to the set of all such elements as the interior grid. Let $\mathcal{N}$ denote the set of six nearest neighbors at the distance h from x, possibly including some points outside of the snowflake. Then, as shown in [2], the Laplacian of a function u at x can be approximated using a difference quotient:

$$\triangle u(x) \approx \frac{2}{3h^2}\left(\left(\sum_{y\in\mathcal{N}} u(y)\right) - 6u(x)\right). \tag{1}$$

Note that Formula 1 states that the Laplacian of u at an interior grid point can be approximated with a linear combination of nearby values of u. We would like to eliminate reference to the exterior grid points, called ghost points in [3], and we must enforce the boundary condition. If we view this grid as a graph, then each ghost point is adjacent to precisely one interior grid point. (Recall that multiple ghost points can occur at the same location.) Now let x be a ghost point and let x' be the interior point adjacent to x. Symmetry properties of the Laplacian imply that boundary conditions can be enforced by setting $u(x) = -u(x')$.

We can now write a version of Formula 1 that does not refer to the ghost points. Let x be an interior grid point, let N' denote its set of nearest interior neighbors (not including ghost points), and let $n' = \#(N')$. Then (as described in [3]), since each ghost point contributes $-u(x)$ to the sum in Formula 1, the approximation may be rewritten

$$\triangle u(x) \approx \frac{2}{3h^2}\left(\left(\sum_{y\in\mathcal{N}'} u(y)\right) - (12 - n')\, u(x)\right). \tag{2}$$

Equation 2 again expresses $\triangle u(x)$ as a linear combination of nearby values of u and uses only interior grid points. Thus, Equation 2 may be described as multiplication of the vector of values of u at the interior grid points by the appropriate matrix.

This particular grid was introduced in [3]. Earlier studies, such as [2], used a similar but finer grid which incorporated the vertices of the approximation of the boundary. It is then natural to enforce the boundary condition by simply requiring that $u(x) = 0$ at these vertices. The authors of [3] report higher accuracy using a larger spacing, thus the technique here might be thought of as more efficient.

3 Generating images

Our primary purpose is to explain the process of image generation using the mathematical ideas outlined in the previous section. The images in this paper were all generated using Mathematica 6.0. An earlier paper [4] presents the details of the Mathematica code. That paper uses Mathematica 5.2 while the current paper uses some enhancements in version 6.0 to generate much nicer images. We will comment on specific enhancements as we present the images.

The first step to generate vibrational images is to write down the coefficient matrix determined by the set of equations 2. This is called the Laplacian matrix. The rows and columns of the Laplacian matrix are indexed by the grid nodes inside the snowflake in Figure 2. Thus, we assume that these nodes are numbered. The nodes in the first level approximation in Figure 3 are explicitly numbered, while the shading of the nodes in the second level approximation indicates the number. Using this numbering scheme, we can write down the Laplacian matrix. The level 1 Laplacian matrix is the following 13×13 matrix.

$$\begin{pmatrix}
-12 & 2 & 2 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -14 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 \\
2 & 2 & -14 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\
2 & 0 & 2 & -14 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 \\
2 & 0 & 0 & 2 & -14 & 2 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\
2 & 0 & 0 & 0 & 2 & -14 & 2 & 2 & 0 & 0 & 0 & 0 & 2 \\
2 & 2 & 0 & 0 & 0 & 2 & -14 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & -20 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & -20 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -20 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -20 & 0 & 0 \\
0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -20 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -20
\end{pmatrix}$$

The level 2 Laplacian matrix has dimension 133, but its structure can be easily visualized by dividing a square into 133 by 133 sub-squares and shading them according to the magnitude of the corresponding term in the Laplacian matrix. This is illustrated in Figure 4.

Now that we have the Laplacian matrix, we can approximate the eigenvalues of the Laplacian on the Koch snowflake using the eigenvalues of the matrix. The eigenfunctions on the snowflake can be approximated with the eigenvectors of the matrix. The i^{th} term in an eigenvector indicates the value of an eigenfunction at the i^{th} node in the interior grid. Furthermore, the interior grid naturally decomposes the snowflake into a collection of (mostly) triangles and quadrilaterals. We can now use values in the eigenvector together with the polygonal decomposition of the Koch snowflake to set up a three-dimensional image. Two such images are shown in Figure 5 and Figure 6.

These images would not have been possible with version 5.2. Figure 5 of [3] and the first figure in section 2.3 of [4] are very similar to Figure 6, but were generated with an earlier version of Mathematica. Those images are much more faceted. The reason is that V6 allows the specification of vertex normals to smooth the surface.

Alternatively, we can generate a two-dimensional shaded contour plot of the eigenfunction. We simply color each node in the interior grid according to the value of the eigenvector at that point and interpolate throughout the triangles. (Note that version 5.2 of Mathematica did not allow interpolation of color throughout a triangle.) We can also draw contours over the images as follows. Fix a z value and for each triangle in the decomposition of the snowflake we use the vertex values to determine if there are two points on the edge of that triangle where the value of the eigenfunction is z. If there are two such points, we draw the line segment between them. This is exactly the strategy used to generate Figures 7 and 8.

Finally, we can combine the two strategies used to generate Figure 6 and Figure 7. This leads to the image in Figure 9. A higher mode of vibration is shown in Figure 10. In these images, the use of vertex normals to smooth the surface is even more important than in Figures 5 and 6.

References

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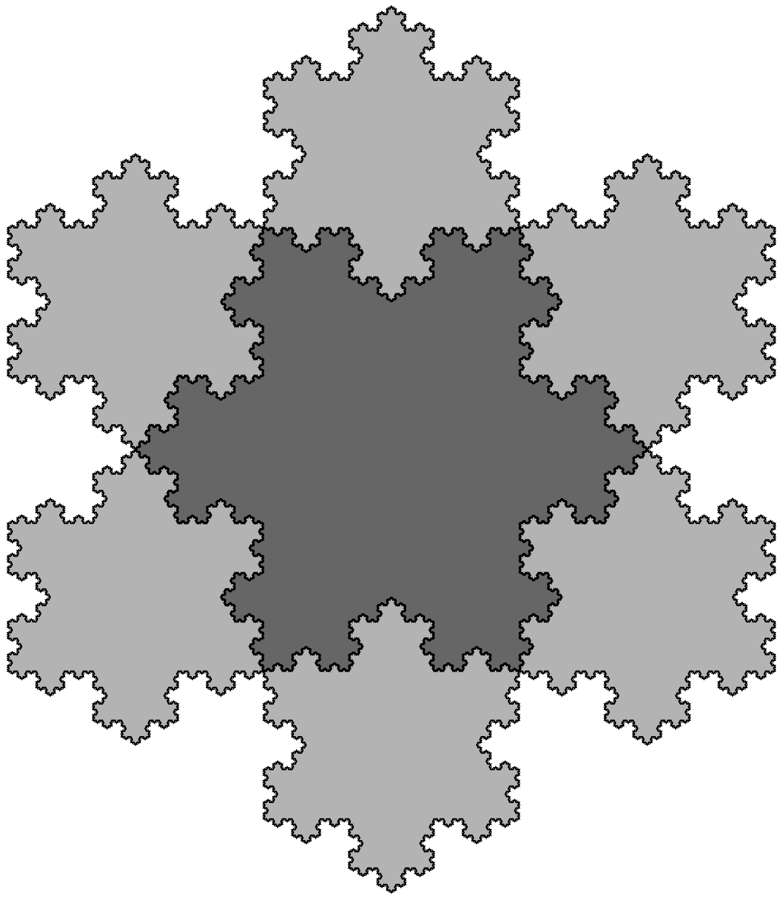

Fig. 1. Self-similar decomposition of the Koch snowflake

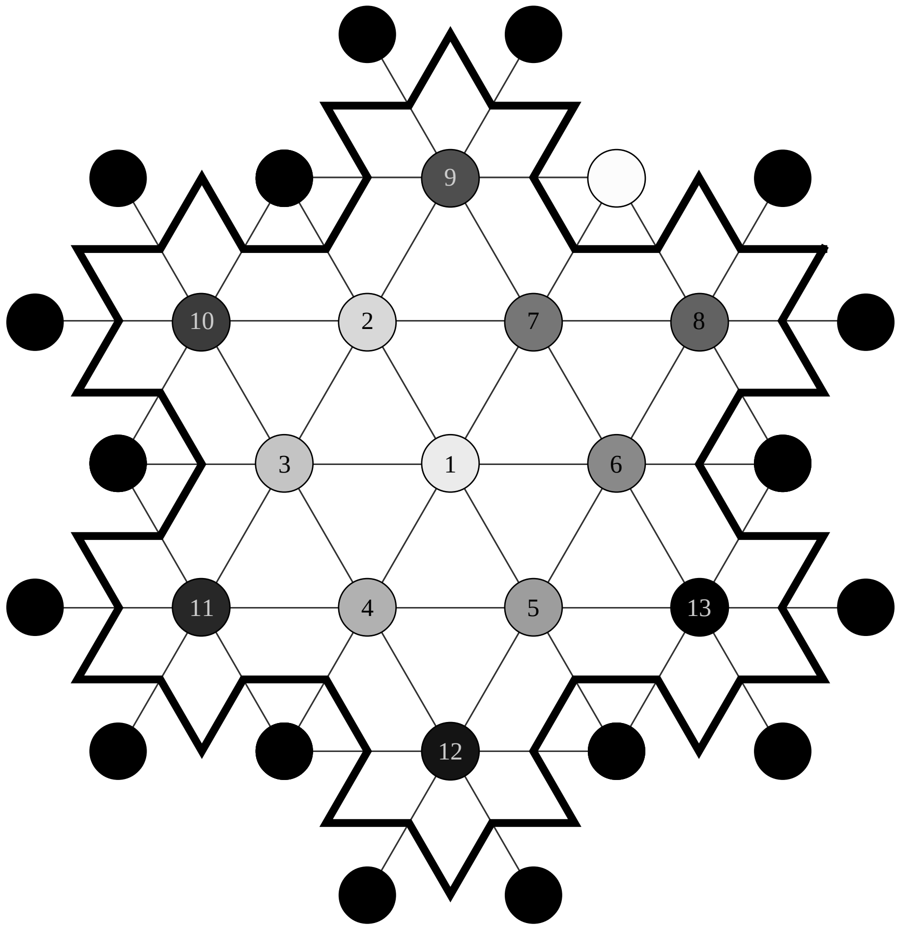

Fig. 2. Level 1 approximation to the Koch snowflake with a grid

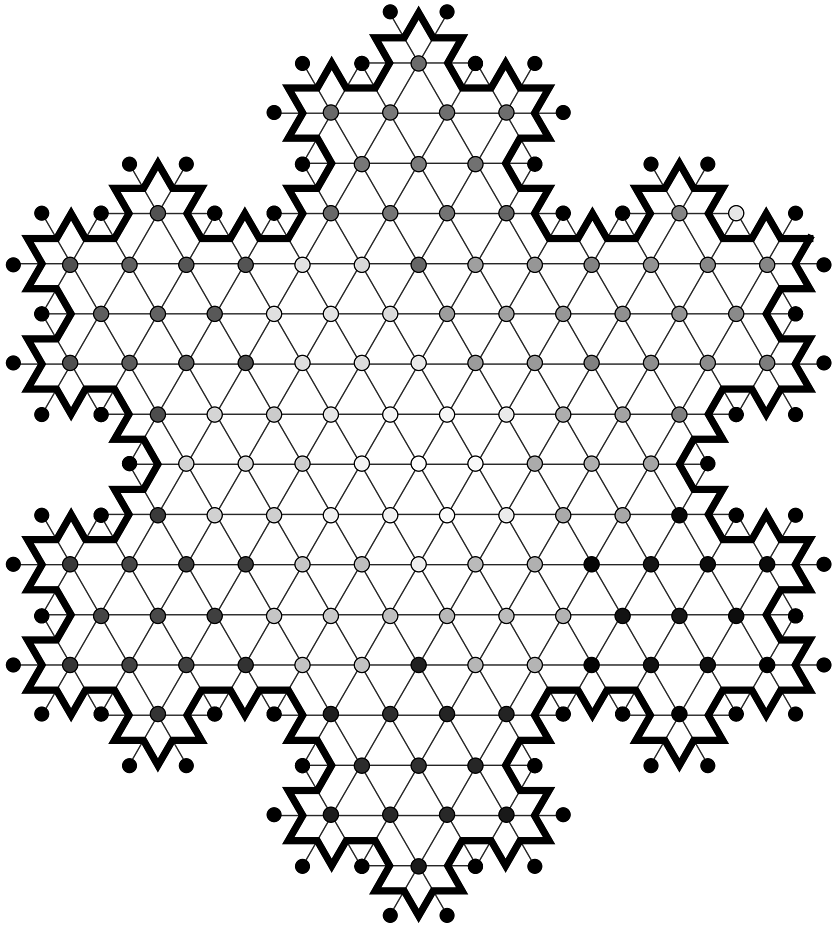

Fig. 3. Level 2 approximation to the Koch snowflake with a grid

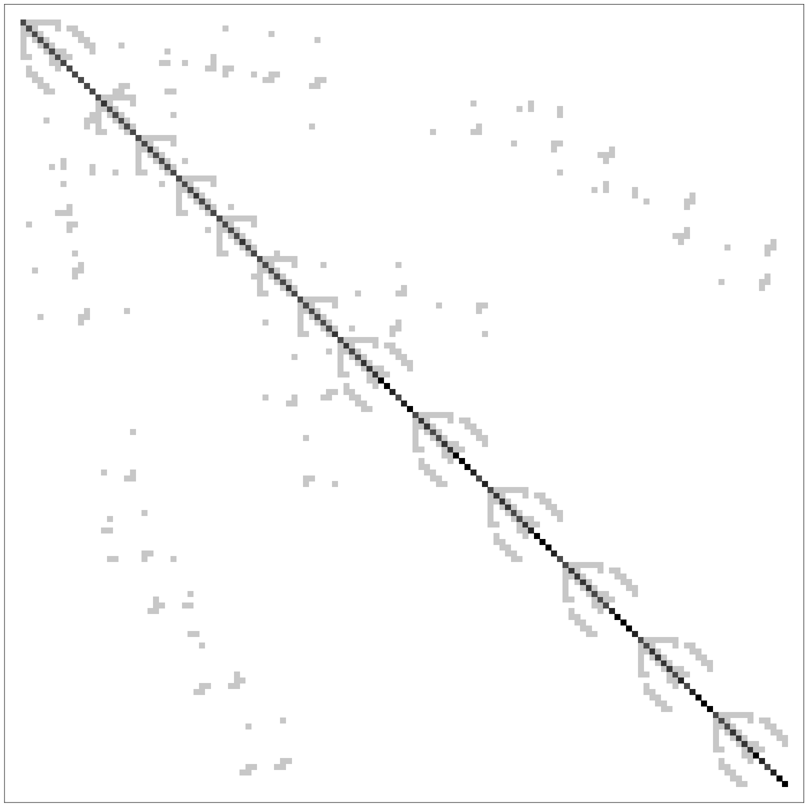

Fig. 4. Structure of the level 2 Laplacian matrix

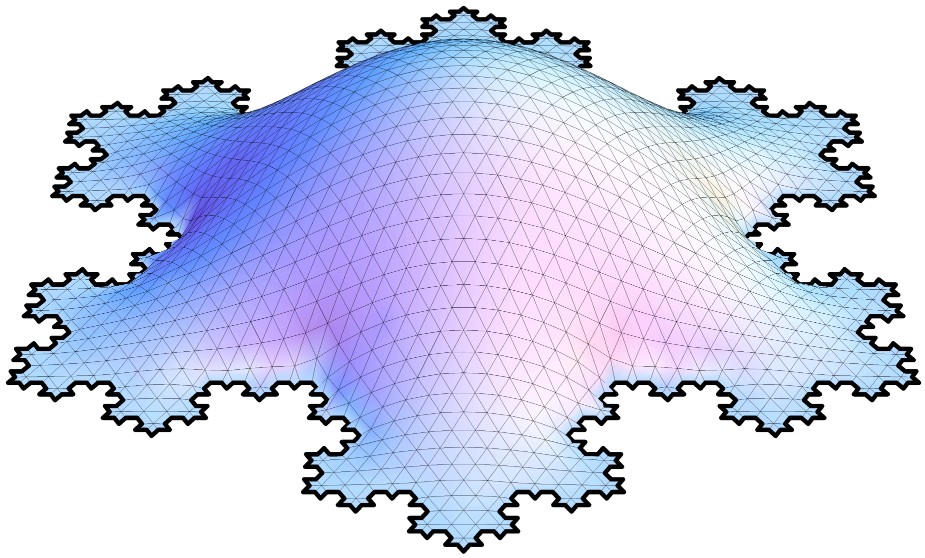

Fig. 5. The lowest frequency mode of vibration

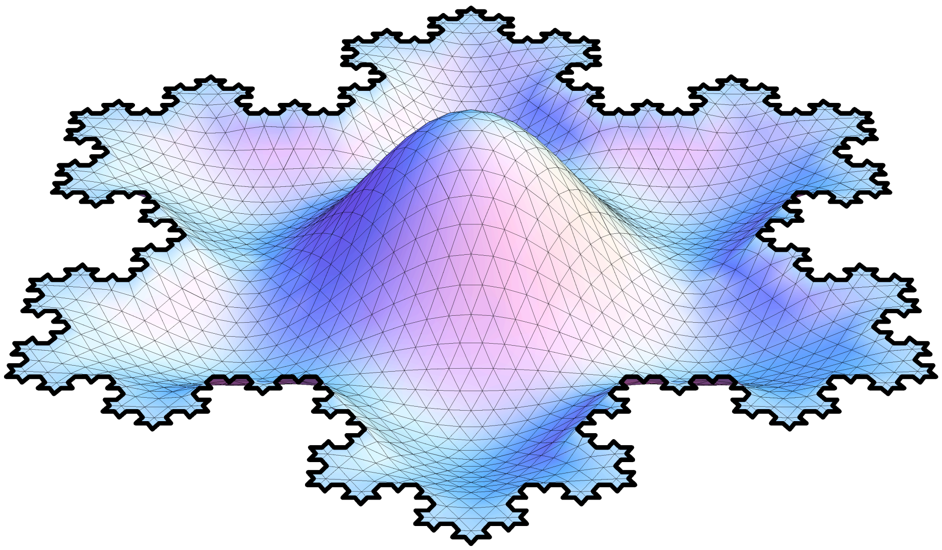

Fig. 6. The mode of vibration corresponding to the sixth smallest eigenvalue

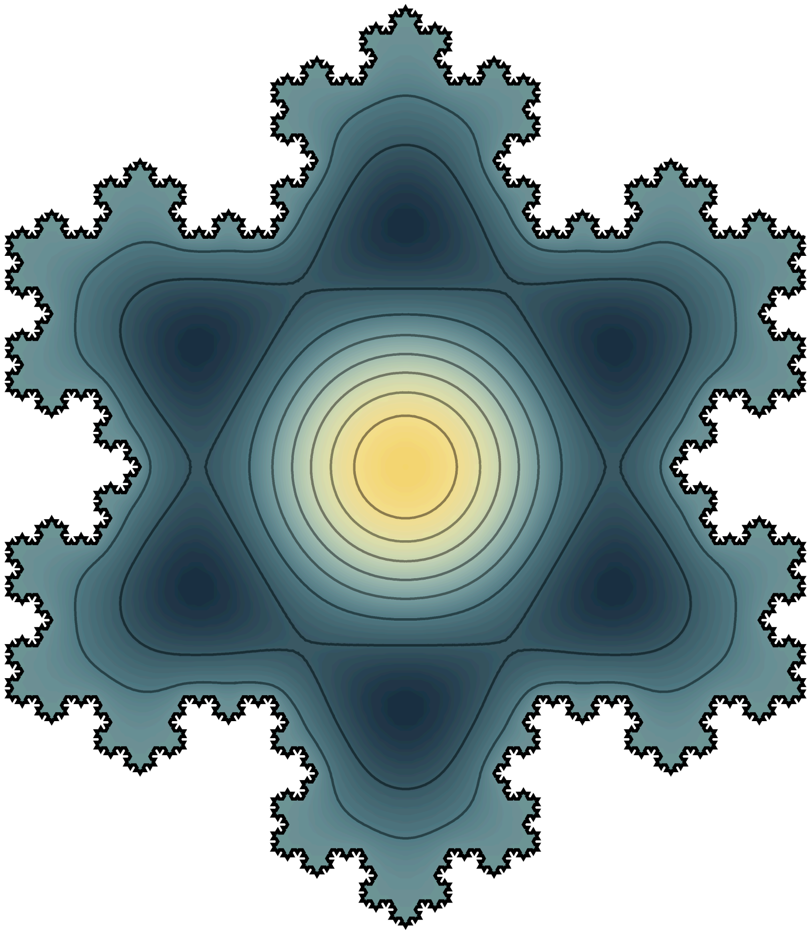

Fig. 7. A contour plot of the sixth mode of vibration

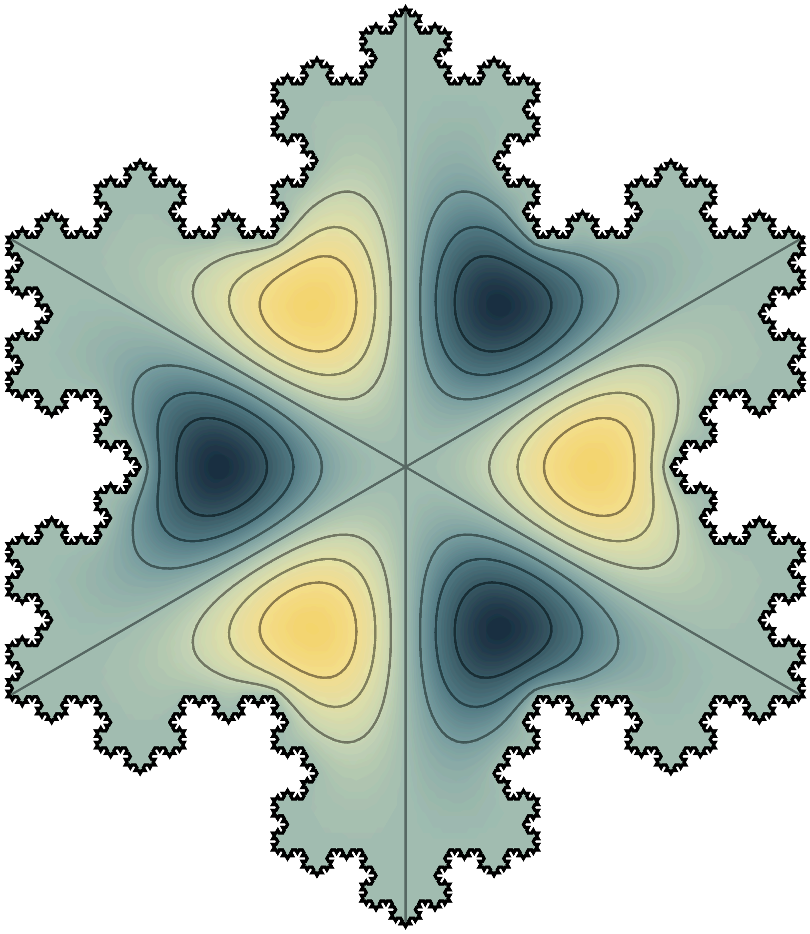

Fig. 8. A contour plot of the tenth mode of vibration

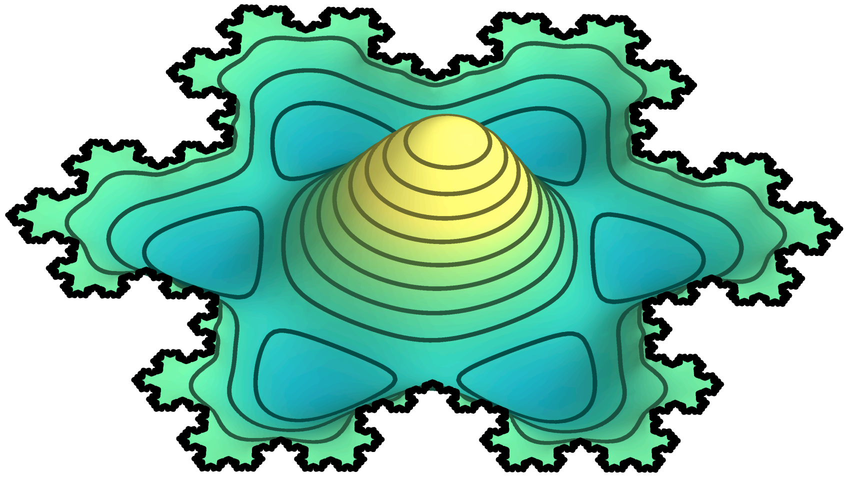

Fig. 9. A 3D plot with contours of the sixth mode of vibration

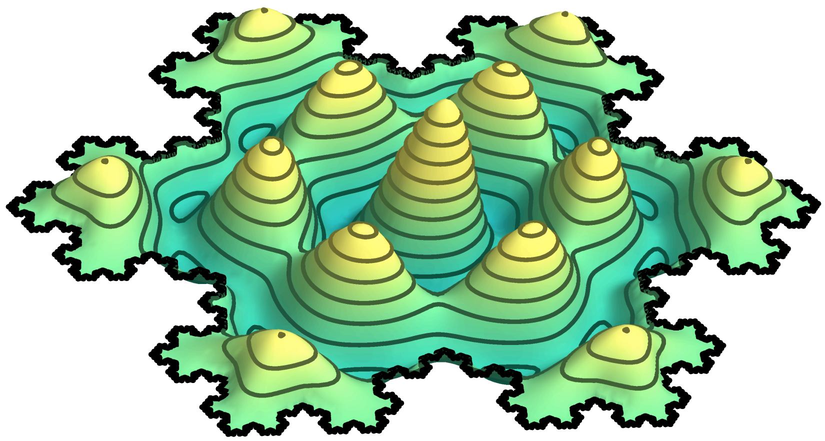

Fig. 10. A 3D plot with contours of the 38^{th} mode of vibration