Digraph Self-Similar Sets and Aperiodic Tilings

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1 Introduction

Self-similarity is a concept often associated with fractal geometry. There are many interesting self-similar sets in the plane which would not generally be considered fractal, however (although their boundaries might be fractal). Such sets provide a fresh way of looking at tilings of the plane. Furthermore, a generalization of self-similarity, called digraph self-similarity, provides a way to construct aperiodic tilings.

2 Self-similarity and tiling

A set which is composed of several scaled images of itself may be thought of as *self-similar* as described in [5] and [6]. (We will write a more mathematical definition shortly.) A square is a simple example of a self-similar set in the plane, being composed of four copies of itself, scaled by the factor $\frac{1}{2}$. Each of these four copies are in turn composed of smaller copies, etc. By iterating the decomposition and scaling up, we generate an obvious checkerboard tiling of the plane. A *tiling* is simply a family of closed sets which cover the plane and intersect only in their boundaries. A tremendous amount of information on tilings may be found in [8].

The connection with self-similarity suggests the possibility of introducing fracticality into the picture. Figure 1 illustrates this with a set called the terdragon, which is composed of three copies of itself all scaled by the factor $\frac{1}{\sqrt{3}}$. A tiling using the terdragon is illustrated in figure 2. Sets with fractal boundaries which tile the plane are called "fractiles" in [5]. General techniques for creating self-similar tiles are discussed in [2] and [4].

3 Iterated function systems

A precise description of self-similarity may be stated using an *iterated function system*, or IFS. If r is a positive real number, a *similarity* with ratio r is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $|f(x) - f(y)| = r|x - y|$ for all $x, y \in \mathbb{R}^2$. If $r < 1$, the similarity is called *contractive*. An iterated function system is a finite collection of contractive similarities $\{f_i\}_{i=1}^m$. For any IFS, there is a unique non-empty, closed, bounded subset E of $\mathbb{R}^2$ such that

$$E = \bigcup_{i=1}^{m} f_i(E).$$

The set E is called the *invariant set* of the IFS and sets constructed in this manner are also called self-similar.

Iterated function systems are easily described using matrix representations. Any contractive similarity f_i may be expressed in the form $A\vec{x} + \vec{b}$, where A is a matrix and $\vec{b}$ is a translation vector. A rotation about the origin through angle θ may be represented using the matrix

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Dividing the matrix through by r achieves the desired contractivity factor. For example, the following list of functions defines the IFS for the terdragon.

$$\begin{aligned} f_1(\vec{x}) &= \frac{1}{\sqrt{3}} R\left(-\frac{\pi}{6}\right)\vec{x} \\ f_2(\vec{x}) &= \frac{1}{\sqrt{3}} R\left(\frac{\pi}{2}\right)\vec{x} + \begin{pmatrix} 1/2 \\ -1/(2\sqrt{3}) \end{pmatrix} \\ f_3(\vec{x}) &= \frac{1}{\sqrt{3}} R\left(-\frac{\pi}{6}\right)\vec{x} + \begin{pmatrix} 1/2 \\ 1/(2\sqrt{3}) \end{pmatrix} \end{aligned}$$

4 Digraph self-similarity and aperiodic tilings

Figure 3 illustrates a generalization of self-similarity called *digraph self-similarity* or *mixed self-similarity*. Digraph self-similarity was introduced in [9] and also described in [5]. The terminology mixed self-similarity was introduced in [1] to describe the same idea. We will use the terminology of [5]. Digraph self-similarity is exhibited by a collection of sets, each of which is composed of scaled images chosen from the collection. In figure 3 for example, the type A triangle is composed of two copies of itself together one copy of the type B triangle. The type B triangle is composed of one copy of itself together with one copy of the type A triangle. The scaling factor for all images is $\frac{1}{\tau}$ where τ is the golden ratio.

As with self-similar sets, the basic decomposition may be iterated to obtain tilings of the plane. In figure 4, we see the fourth step in the decomposition.

5 Digraph iterated function systems

Any collection of digraph self-similar sets can be described using a *directed-graph iterated function system*, or digraph IFS. A digraph IFS consists of a directed multigraph G together with a contractive similarity f_e from $\mathbb{R}^2$ to $\mathbb{R}^2$ associated with each edge of G. A directed multigraph consists of a finite set V of vertices and a finite set E of directed edges between vertices. Given two vertices, u and v, we denote the set of all edges from u to v by E_{uv}. Given a digraph IFS, there is a unique set of non-empty, closed, bounded sets K_v, one for each $v \in V$, such that for every $u \in V$

$$K_u = \bigcup_{v \in V,\, e \in E_{uv}} f_e(K_v).$$

Sets constructed using a digraph IFS are said to exhibit *digraph self-similarity.*

The digraph IFS for the type A and B triangles is shown in figure 5. The labels on the edges correspond to the following similarities mapping one triangle to part of another (perhaps the same) triangle. Note that these similarities involve reflections as well as rotations.

$$\begin{aligned}
a_1(\vec{x}) &= \frac{1}{\tau} R\left(\frac{3\pi}{5}\right)\vec{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
a_2(\vec{x}) &= \frac{1}{\tau} R\left(\frac{4\pi}{5}\right)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} \cos(\frac{2\pi}{5}) \\ \sin(\frac{2\pi}{5}) \end{pmatrix} \\
a_3(\vec{x}) &= \frac{1}{\tau} R\left(\frac{3\pi}{5}\right)\vec{x} + \begin{pmatrix} \cos(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
b_1(\vec{x}) &= \frac{1}{\tau} R\left(-\frac{3\pi}{5}\right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \vec{x} + \begin{pmatrix} \frac{1}{\tau} \\ 0 \end{pmatrix} \\
b_2(\vec{x}) &= \frac{1}{\tau} R\left(-\frac{4\pi}{5}\right) \vec{x} + \begin{pmatrix} \cos(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) \end{pmatrix}
\end{aligned}
$$

6 Penrose tilings

In 1973 and 1974, Roger Penrose discovered several families of sets which tile the plane aperiodically and (if certain matching conditions are enforced) only aperiodically. Introductions to Penrose tilings may be found in [7] and [8]. The most well known of these are tilings by kites and darts. It turns out that this type of tiling is closely related to the digraph self-similar triangular tilings of the previous section.

Figure 6 illustrates the kite and dart. The dotted lines indicate that the kite is union of two type A triangles and the dart is the union of two type B triangles. The filled and unfilled disks at the vertices are used to enforce a matching condition. When tiling the plane with kites and darts, we demand that filled disks meet filled disks and unfilled disks meet unfilled disks. This matching condition guarantees that any tiling by kites and darts will be aperiodic, i.e. no translation of the tiling maps each tile to another tile. Figure 7 shows how we may generate a tiling by kites and darts using the digraph self-similar set strategy by simply deleting certain edges of the initial triangles. Note that we have marked the vertices of the triangles to match the markings of the kite and dart and that the functions a_2 and b_1 from the previous section involve reflections to get deleted edges to line up. Figure 8 shows the tiling after four steps in the iteration.

7 Penrose pentacles and fractal boundaries

Just as with strictly self-similar sets, we can use digraph self-similarity to generate aperiodic tilings by sets with fractal boundaries. The first such example, a modification of the kite and dart tilings, was published in [1]. We present a different such tiling here.

Our aperiodic fractiles will be based on the first aperiodic set of tiles discovered by Penrose. These tiles are called the pentacles and are illustrated in figure 9. All of the sides have length one and the angles are all integer multiples of $\frac{\pi}{5}$. The labels indicate a matching condition, which again assures aperiodicity. The edges labeled 0 must fit against edges labeled $\bar{0}$, 1 against $\bar{1}$, and 2 against $\bar{2}$. Note that the three pentagons are congruent, but have different matching conditions. A portion of a tiling by pentacles is shown in figure 10.

In Penrose's analysis, he indicates that these sets may be assembled into "patches" approximately resembling the original tiles as illustrated in figure 11. If we iterate his patchwork, we generate tilings such as the one illustrated in figure 10. Penrose goes through an additional step at each iteration, however, to eliminate the fractal boundary. From our viewpoint, it is more natural to embrace the fractal nature of the boundary. Fractal boundary versions of the pentacles are shown in figure 12. We see how these sets fit together to form a collection of digraph self-similar sets in figure 13. The digraph IFS for these sets is shown in figure 14. The digraph has been abbreviated by collapsing all edges between two vertices and labelling them with a list of functions. For example, the function s_1 maps the red star-like shape in the upper left of figure 12 onto the red sub-shape in the upper left of figure 13. The function s_1 is defined by

$$s_1(\vec{x}) = \frac{1}{\tau^2} R(\pi)\vec{x},$$

where $R(\theta)$ represents the matrix which rotates through the angle θ. The digraph IFS is composed of 42 functions in all. Part of an aperiodic tiling by these sets is shown in figure 15.

8 Implementation Notes

All of the images in this paper were generated by the **DigraphFractals** *Mathematica* package written by the author and described in [10]. All the code for these images and more examples are available at the author's web page:
http://www.unca.edu/˜mcmcclur/mathematicaGraphics/DigraphFractiles.

References

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[10] McClure, M., "Directed-graph iterated function systems." *Mathematica in Education and Research* **9** (2000), no. 2.

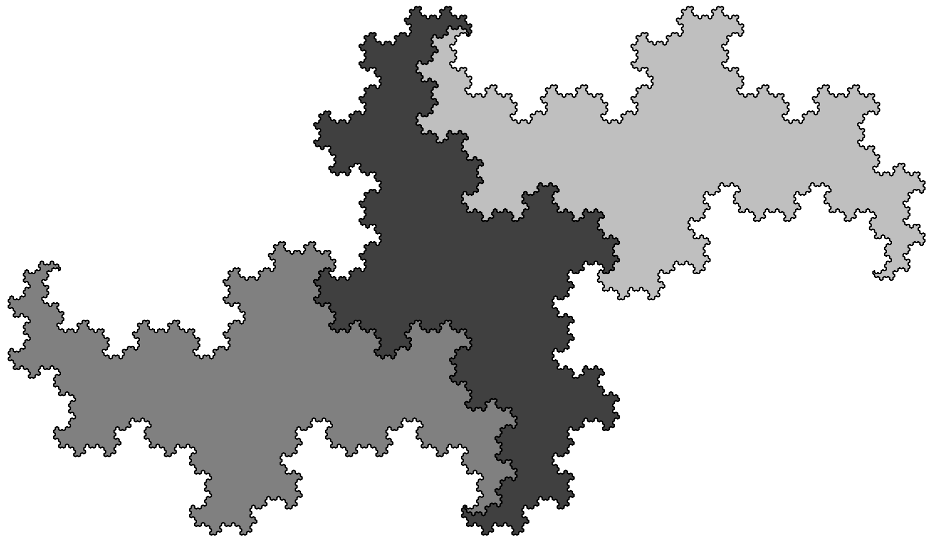

Figure 1: The terdragon

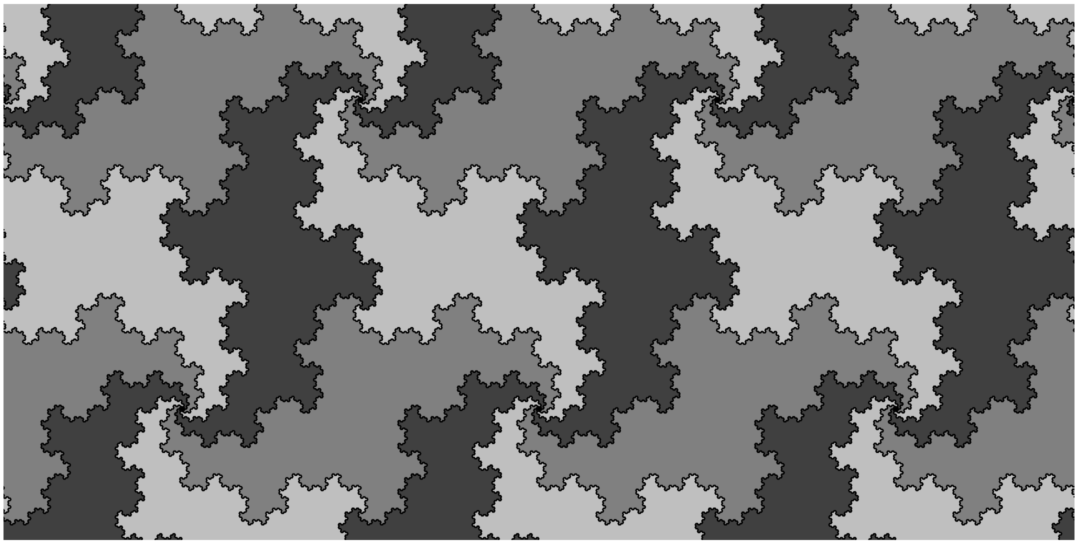

Figure 2: A tiling using terdragons

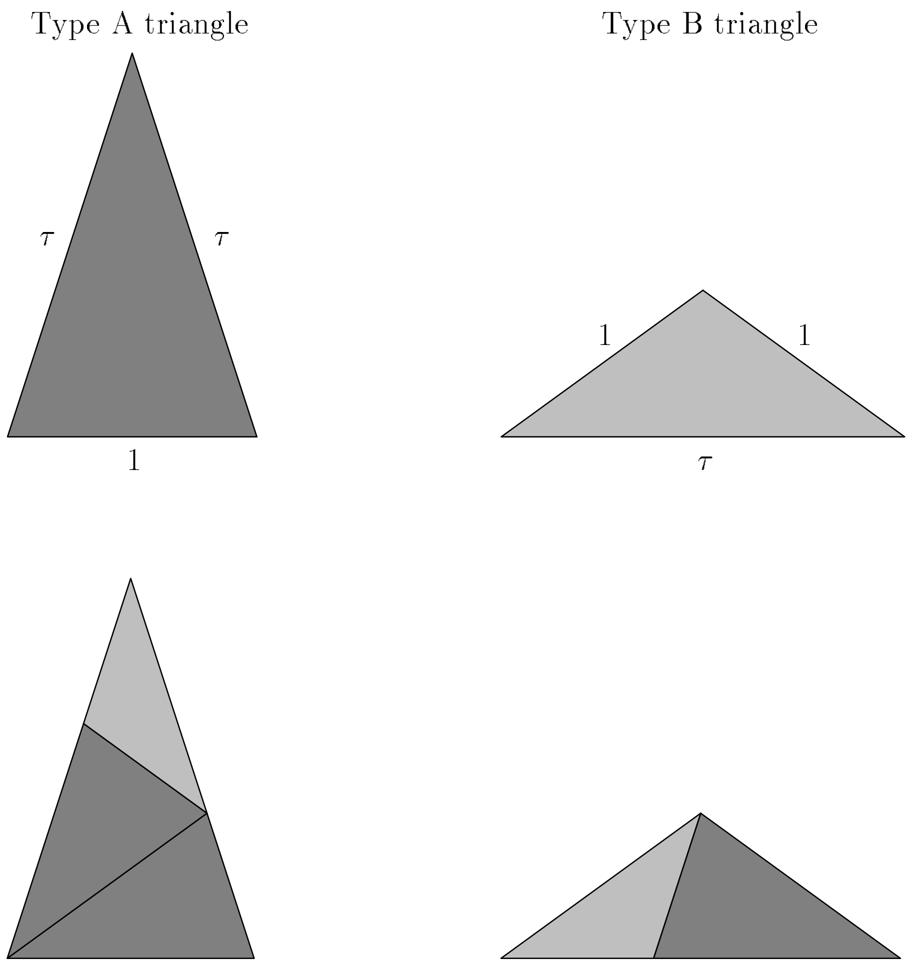

Figure 3: A digraph pair of triangles

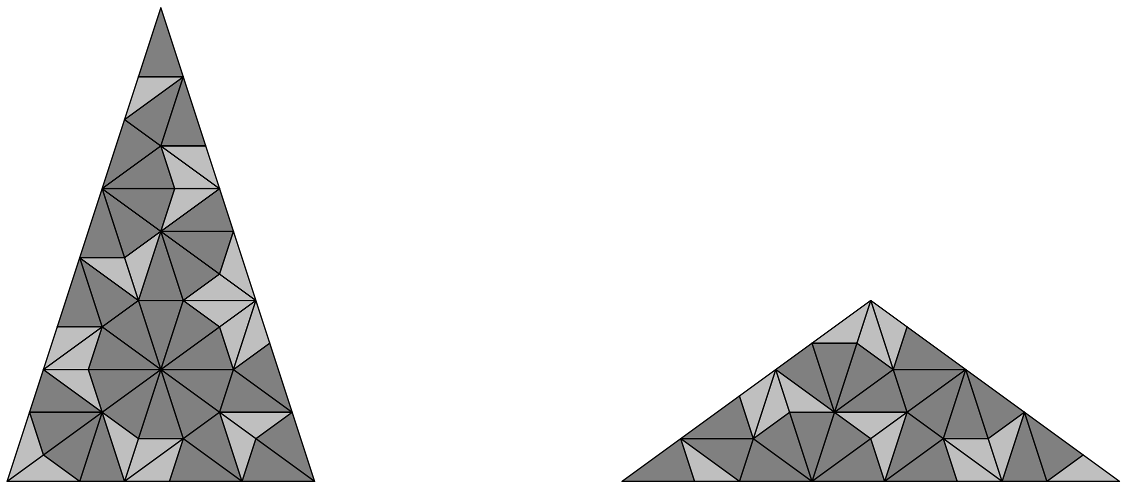

Figure 4: Generating a tiling with the digraph triangles

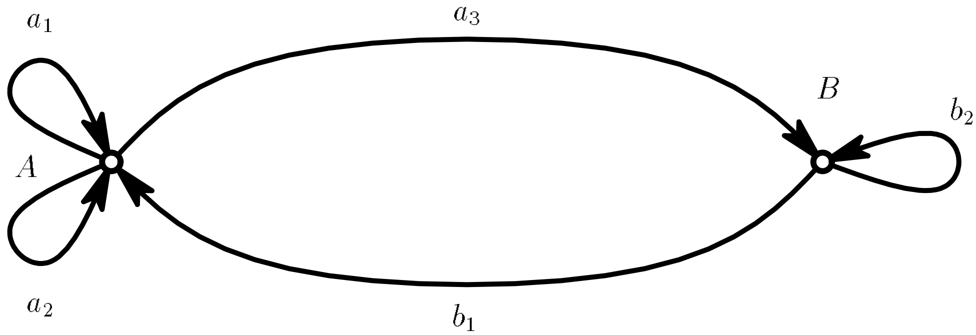

Figure 5: The digraph IFS for the triangles

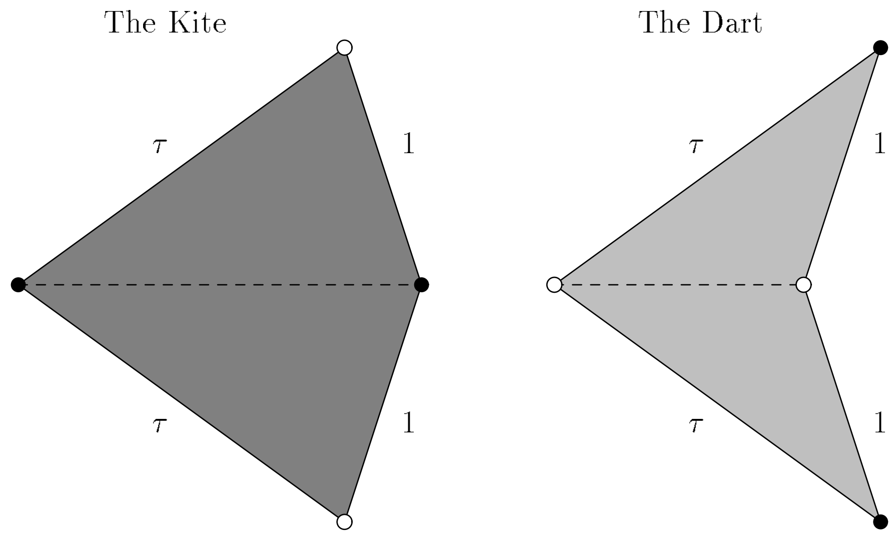

Figure 6: Penrose's kite and dart

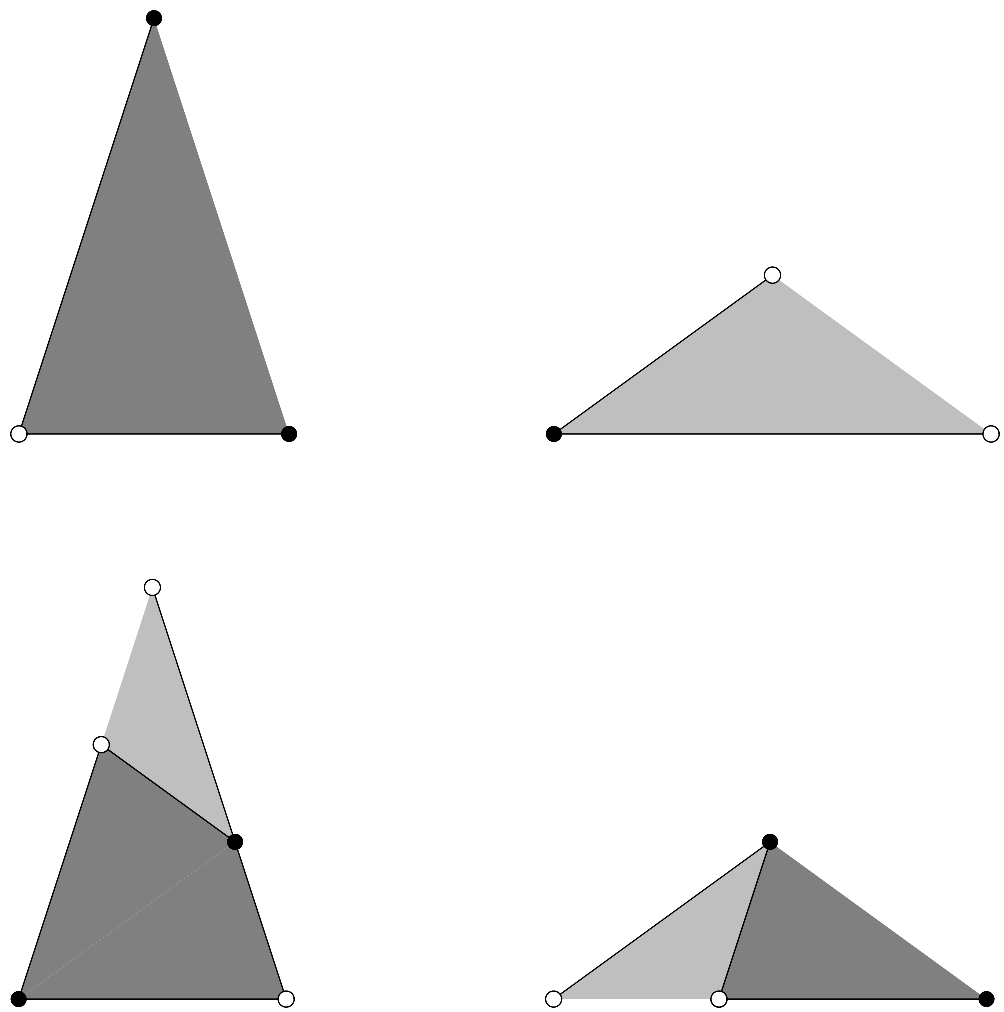

Figure 7: An alteration of the digraph triangles

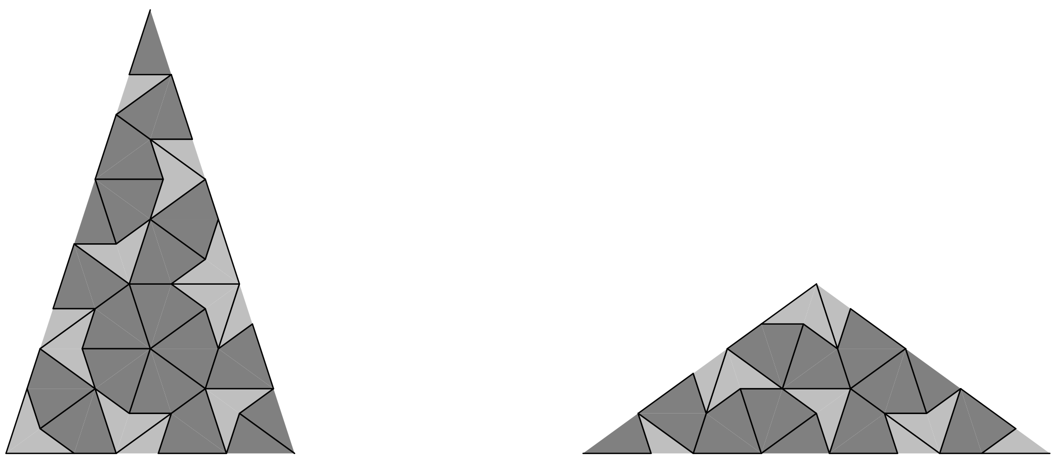

Figure 8: Generating a tiling by kites and darts

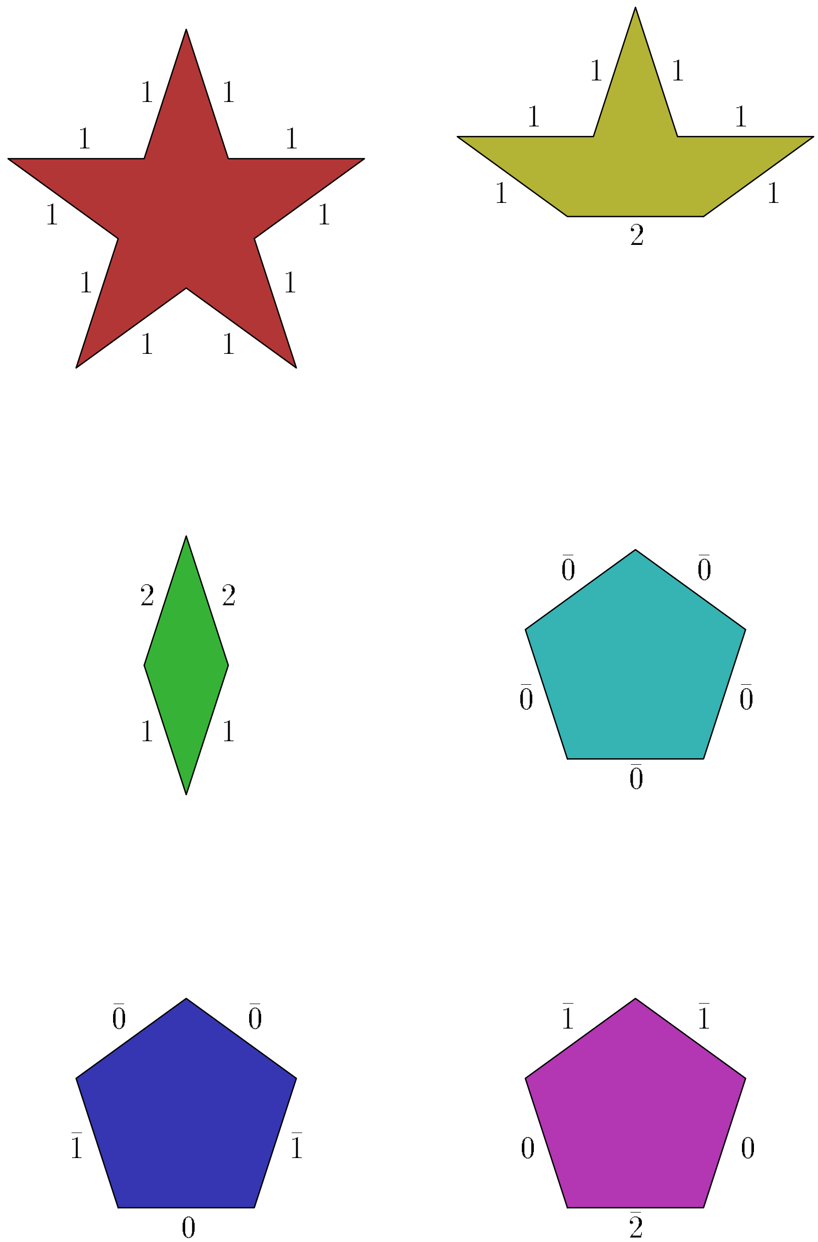

Figure 9: Penrose's Pentacles

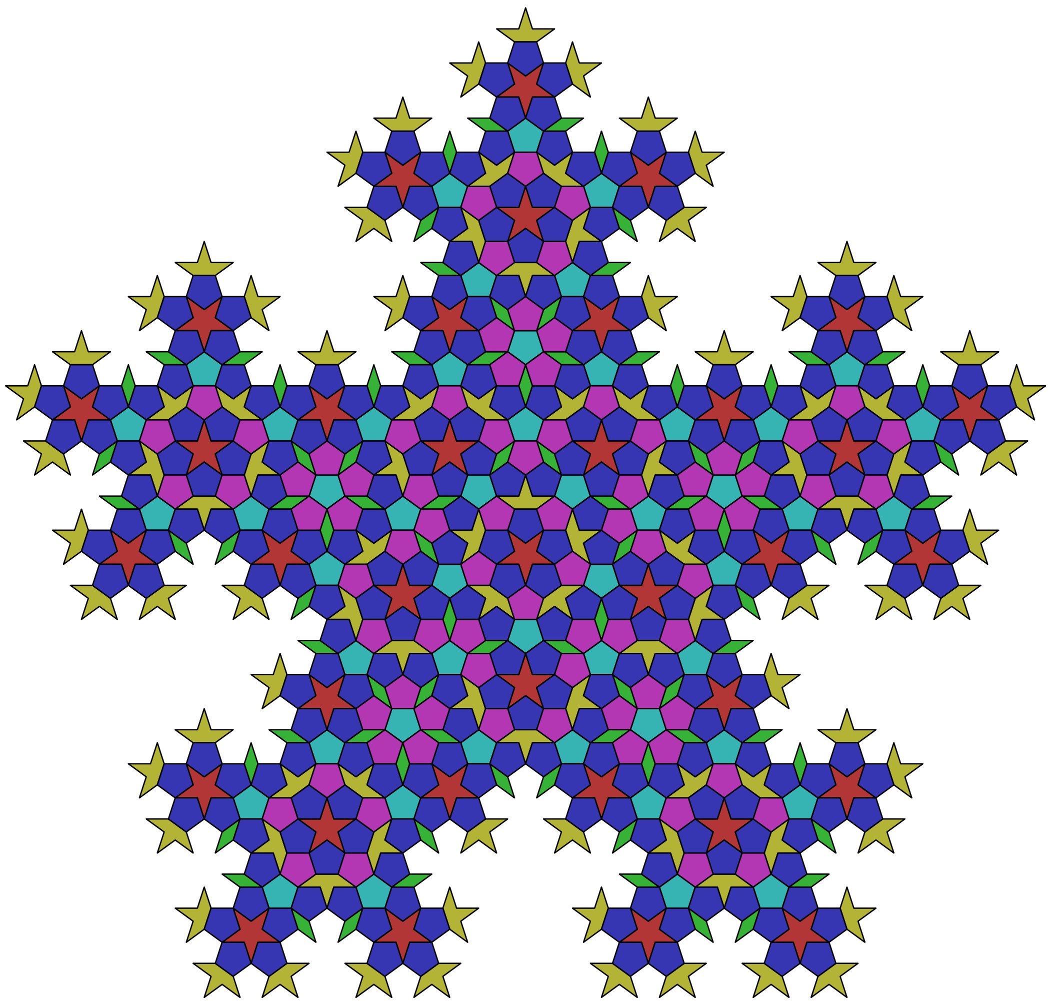

Figure 10: Part of a tiling by pentacles

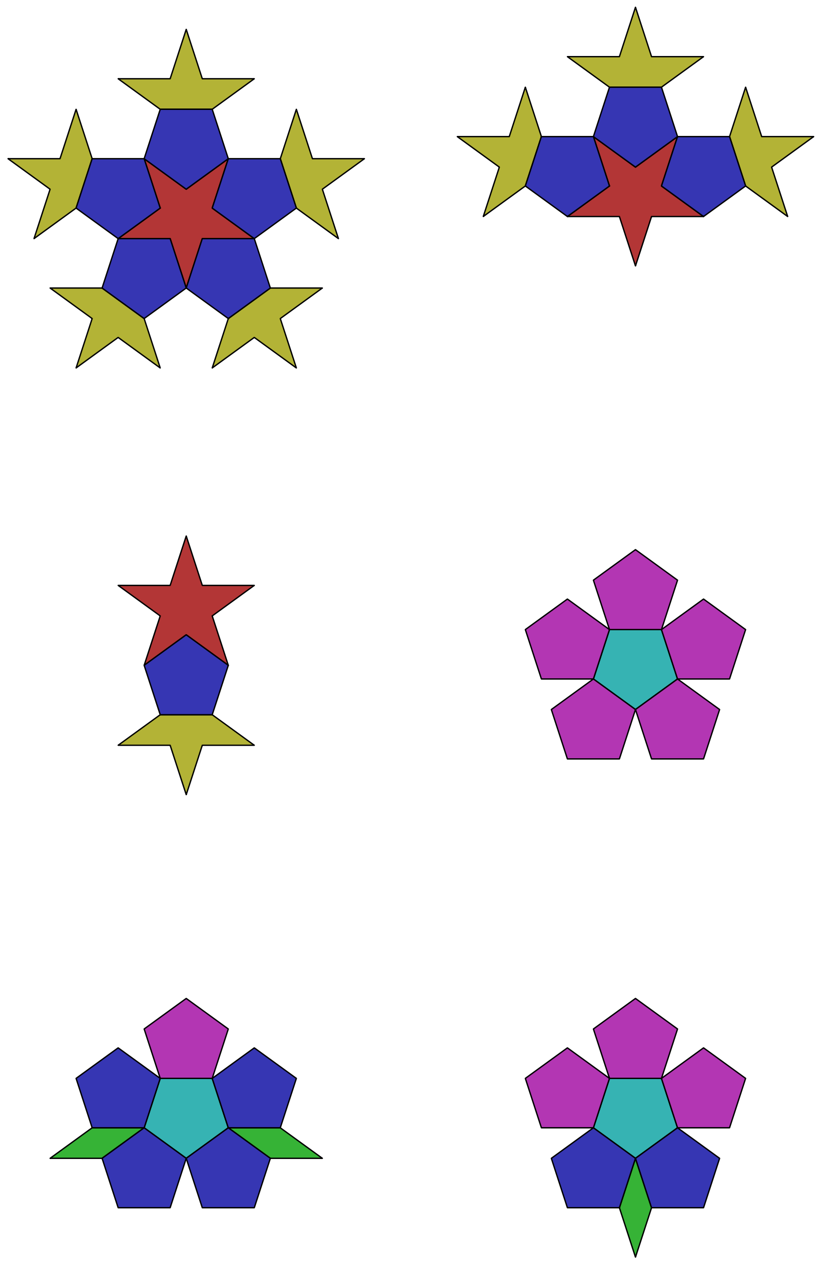

Figure 11: Penrose's patches of pentacles

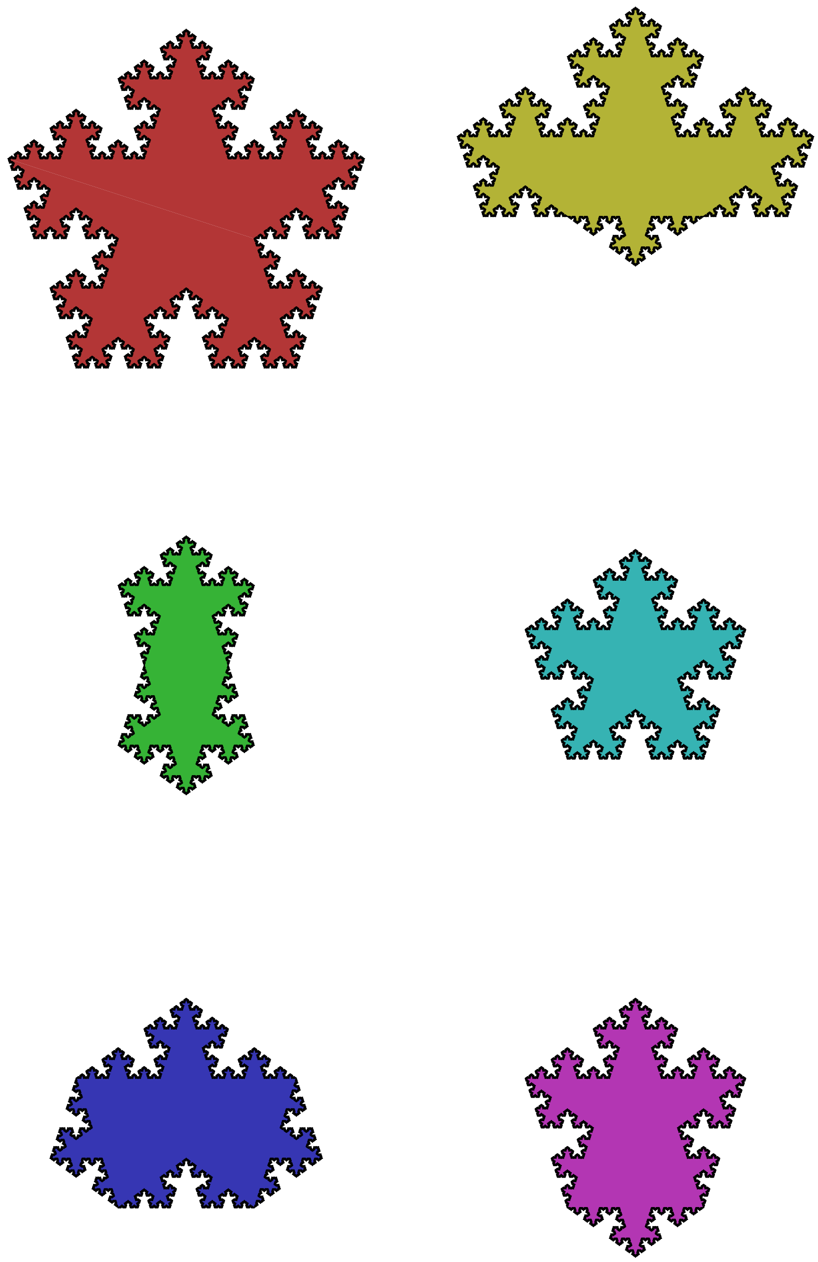

Figure 12: Versions of the pentacles with fractal boundaries

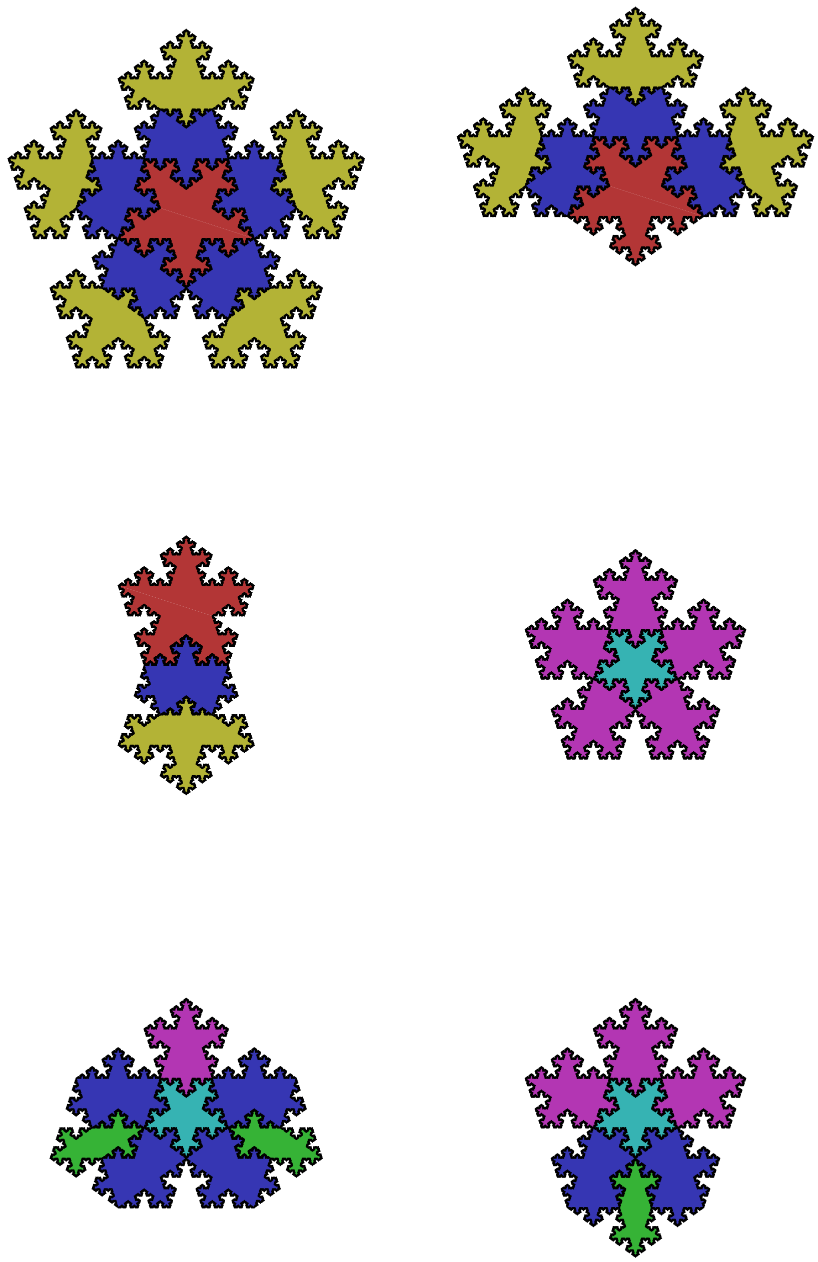

Figure 13: Fitting the pieces together

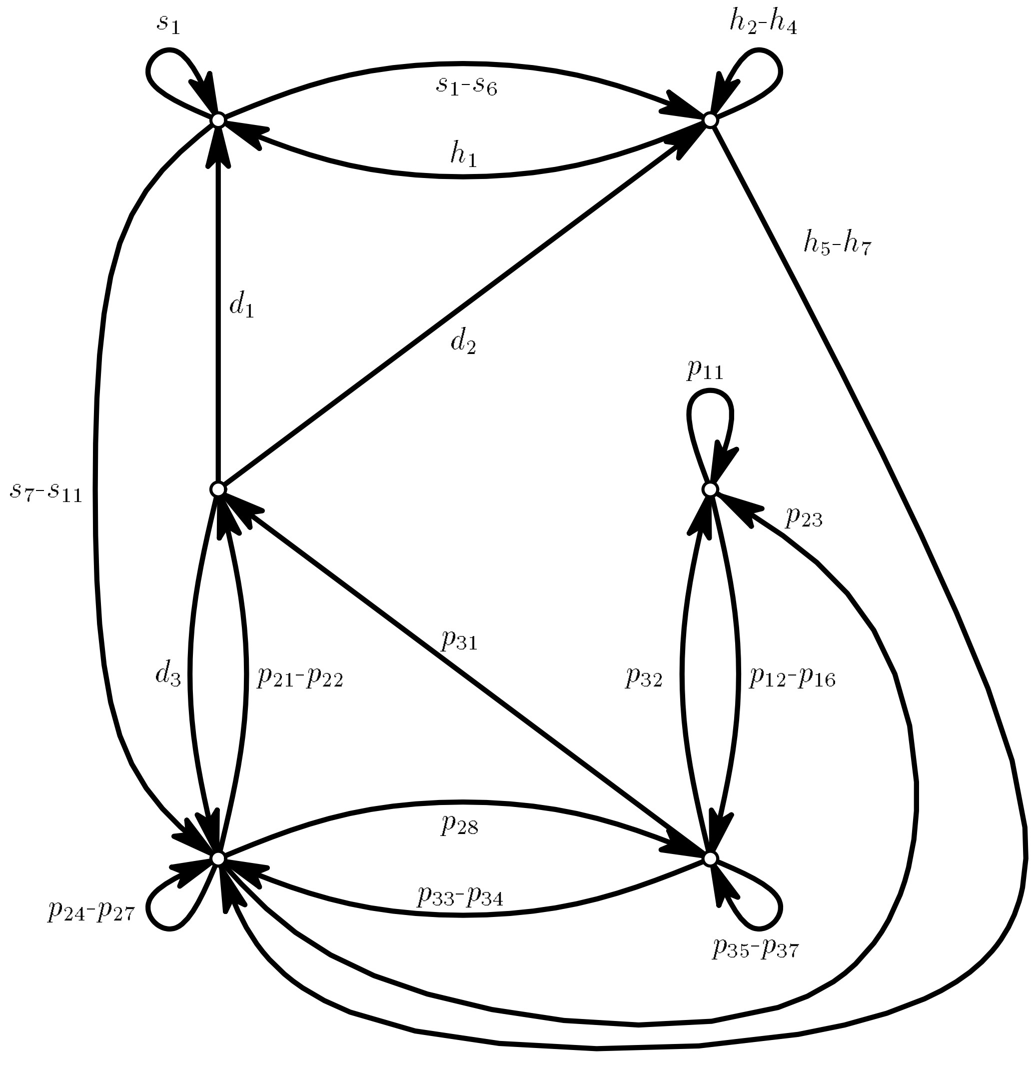

Figure 14: The digraph IFS for the fractal boundary pentacles

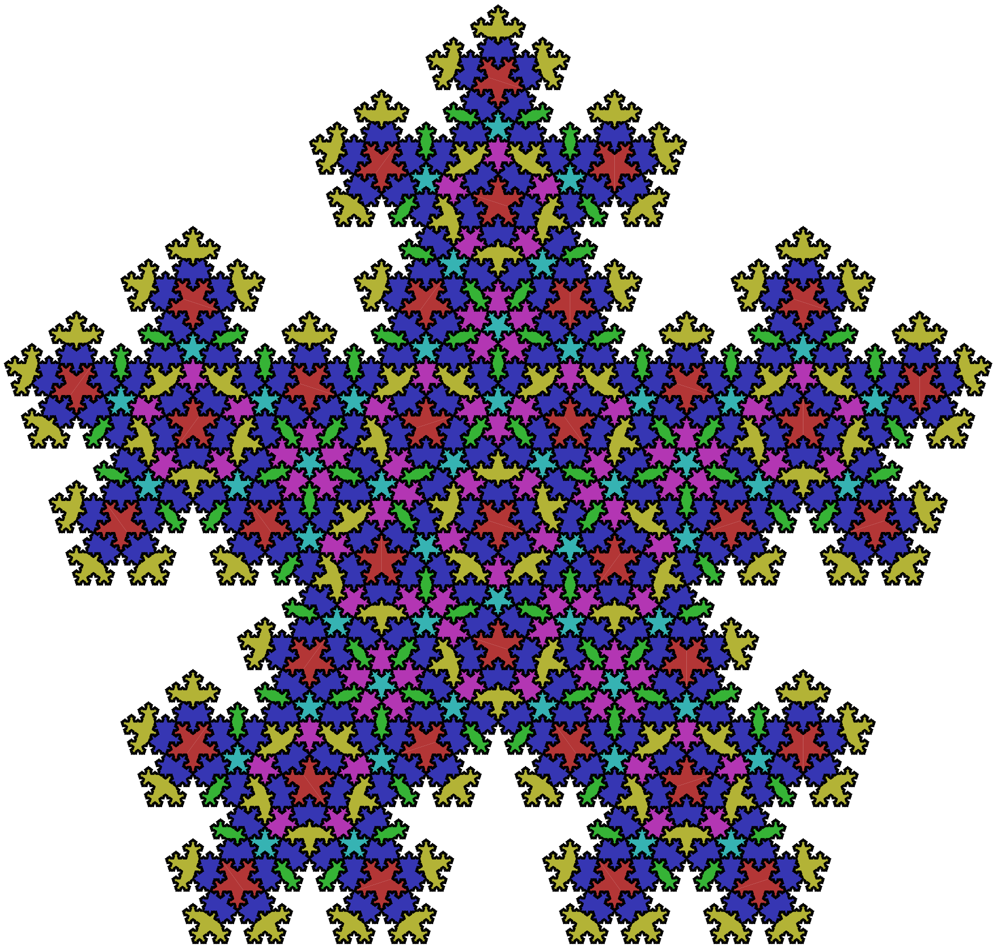

Figure 15: Part of a tiling using the pentacles with fractal boundaries