Fractal Measures on Infinite Dimensional Sets

DISSERTATION

Presented in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy in the Graduate School of The Ohio State University

By

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1994

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CHAPTER I

Introduction

The oldest notion of dimension was clearly intuitive in nature. Mathematicians felt that the line $\mathbb{R}$ was somehow different from the plane $\mathbb{R}^2$ long before they were proven to be non-homeomorphic. Intuition is less useful a guide for more complicated sets, however. Cantor type sets illustrate this nicely. In some sense they are small being able to fill up no part of $\mathbb{R}$. On the other hand, they seem big being uncountable and measure theoretically rich. Thus the need arises for a rigorous definition of dimension.

There are now many such definitions. Fractal dimensions are those that concentrate on the metric structure of a set and lead to the possibility of a set having a non-integral dimension. Thus a fractal dimension allows for the possibility of distinction between sets with equal topological dimension, which is necessarily integral. Cantor's ternary set for example has topological dimension 0, but fractal dimension $\frac{\log 2}{\log 3} \approx 0.6309$. (There are different definitions of fractal dimension, but I know of none that assign some other dimension to Cantor's set.) Thus a fractal dimension somehow captures the bigness of Cantor's set which the topological dimension misses.

The oldest fractal dimension is Hausdorff's [Ha]. His definition is based on a measure theoretic construction of Carathéodory [Ca]. Carathéodory was interested in defining a measure similar to Lebesgue's to measure the size of a smooth m-dimensional sub-manifold of $\mathbb{R}^n$. In [Ha], Hausdorff shows that this may be extended to a natural s-dimensional measure $\mathcal{H}^s$ for any real $s > 0$. This in turn defines dimension by choosing an appropriate parameter s to measure a given set $E \subset \mathbb{R}^n$.

More generally, the dimension of a set E may be roughly associated with a monotone increasing function $\varphi(t)$ defined for $t \geq 0$ and with $\varphi(0) = 0$. In this case write $\dim(E) \sim \varphi(t)$. To say that E has finite dimension s, is to say that $\dim(E) \sim t^s$. The faster φ vanishes at the origin, the larger the corresponding dimension. It is possible if E is a subset of a large metric space, that $\dim(E) \sim \varphi$, where φ disappears faster at the origin than any power of t. Thus we see that fractal dimensions may be used not only to distinguish between sets of equal integral dimension, but also potentially between sets of infinite dimension. It is on such sets that I will concentrate in this dissertation.

The organization is as follows. In chapter two, I state definitions making the above ideas precise. This chapter is based on much earlier work, but tailored to my purposes. There are many definitions stated, but they are all interrelated and some are essentially equivalent. I prove some comparison theorems which indicate that a fairly complete dimensional picture emerges if we can understand two definitions in particular - the Hausdorff dimension and the upper entropy dimension.

In chapter three, I develop some important computational tools and illustrate their use on some spaces which are relatively easy to deal with called sequence spaces. In section 3.4, sequence spaces are in turn used to define another important tool for calculating Hausdorff dimension - the s-nested packing. Finally, in section 3.5 I discuss an example involving Cartesian-products which illustrates why certain definitions have evolved to their current form.

Chapters four and five form the heart of the original material dealing with hyperspaces and function spaces respectively. These are two of the most natural examples of infinite dimensional spaces. Hyperspaces are metric spaces whose elements are subsets of another metric space. Two early papers concerned with dimensions of hyperspaces are [Boa] and [Bro]. In chapter four, I extend results proven there to other notions of dimension and more general metric spaces.

The fifth and final chapter is based on [KolTi] which summarizes the earliest dimensional estimates specifically aimed at infinite dimensional spaces. The theorems proven in [KolTi] deal with what are now called the entropy indices for various sets of functions and are part of the resolution of some questions arising from Hilbert's thirteenth problem. I extend some of these theorems to other notions of dimension.

CHAPTER II

Basic Definitions

In this dissertation I will be defining various notions of dimension in a general setting. (X, ρ) or simply X will denote an arbitrary separable metric space. I will frequently specify that X is complete. For $E \subset X$, $\mathrm{diam}(E)$ will denote the diameter of E. That is $\mathrm{diam}(E) = \sup_{x,y \in X}\{\rho(x,y)\}$. If $E, F \subset X$, then $\mathrm{dist}(E,F) = \inf_{x \in E, y \in F} \rho(x,y)$ denotes the distance from E to F. Given $x \in X$ and $r > 0$, $B_r(x)$ will denote the closed ball of radius r about the point x. That is $B_r(x) = \{y \in X : \rho(x,y) \leq r\}$.

2.1 Measure Theoretic Definitions of Dimension

2.1.1 Hausdorff Dimension

The first definition of a fractal dimension was Hausdorff's [Ha]. Although he was primarily concerned with subsets of Euclidean space, his definition generalizes readily to σ-totally bounded subsets of an arbitrary metric space. Here I follow the very general approach of Rogers [Rog].

For $\varepsilon > 0$ an *ε-cover* of E will be a countable or finite collection of sets, $E_i \subset X$, so that $E \subset \cup_i E_i$ and $\operatorname{diam}(E_i) \leq \varepsilon$ for every i. Let Φ denote the set of all non-decreasing, continuous functions φ defined on some interval $[0, \delta)$ so that $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Such functions will be called *Hausdorff functions*. Given $\varphi \in \Phi$, define a measure $\mathcal{H}^\varphi$ on X as follows:

$$\mathcal{H}^\varphi_\varepsilon(E) = \inf\left\{\sum_i \varphi(\operatorname{diam}(E_i)) : \{E_i\}_i \text{ is an } \varepsilon\text{-cover of } E\right\},$$

$$\mathcal{H}^\varphi(E) = \lim_{\varepsilon \searrow 0} \mathcal{H}^\varphi_\varepsilon(E).$$

Note that $\mathcal{H}^\varphi_\varepsilon(E)$ increases as ε decreases so that $\mathcal{H}^\varphi(E)$ is well defined, though possibly infinite. In [Rog] it is proven that $\mathcal{H}^\varphi$ is a metric outer measure on X. By a *metric outer measure* I mean that $\mathcal{H}^\varphi$ satisfies $\mathcal{H}^\varphi(E \cup F) = \mathcal{H}^\varphi(E) + \mathcal{H}^\varphi(F)$, whenever $\operatorname{dist}(E, F) > 0$. This implies that all analytic (and in particular all Borel) subsets of X are $\mathcal{H}^\varphi$-measurable. Denote the restriction of $\mathcal{H}^\varphi$ to the $\mathcal{H}^\varphi$-measurable subsets of X also by $\mathcal{H}^\varphi$ and call this the *Hausdorff φ-measure* on X.

The idea behind the Hausdorff dimension is that the value of $\mathcal{H}^\varphi(E)$ is governed by the asymptotic properties of $\varphi(t)$ as $t \searrow 0$ in a way indicative of the dimension of E. For example if μ_n is Lebesgue measure on $\mathbb{R}^n$ and $\psi_\alpha(t) = t^\alpha$, then

$$\mathcal{H}^{\psi_\alpha} = \begin{cases} 0 & \text{if } \alpha > n \\ c_n \mu_n & \text{if } \alpha = n \ (c_n \text{ constant}) \\ \text{is non-}\sigma\text{-finite} & \text{if } \alpha < n. \end{cases}$$

In this dissertation I am primarily concerned with infinite dimensional sets and so need an appropriate definition and method of comparing dimensions. Write:

- $\dim(E) \prec \varphi$ if $\mathcal{H}^\varphi(E) = 0$
- $\dim(E) \preceq \varphi$ if $\mathcal{H}^\varphi(E)$ is σ-finite

- $\dim(E) \succ \varphi$ if $\mathcal{H}^{\varphi}(E)$ is non-σ-finite
- $\dim(E) \succeq \varphi$ if $\mathcal{H}^{\varphi}(E) > 0$
- $\dim(E) \asymp \varphi$ if $\mathcal{H}^{\varphi}(E)$ is positive and σ-finite.

I would like to be able to use these same symbols to compare Hausdorff functions. So write:

- $\varphi \prec \psi$ if $\lim_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} = 0$
- $\varphi \preceq \psi$ if $\limsup_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} < \infty$
- $\varphi \asymp \psi$ if $0 < \liminf_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} \leq \limsup_{t \searrow 0} \frac{\psi(t)}{\varphi(t)} < \infty$.

The following lemma justifies the common use of these symbols.

Lemma 2.1.1 If $\dim(E) \preceq \varphi \prec \psi$, then $\dim(E) \prec \psi$.

Proof: Assume first that $\mathcal{H}^{\varphi}(E) < M < \infty$. Let $\delta > 0$, choose $\varepsilon > 0$ such that $0 < t \leq \varepsilon$ implies $\frac{\psi(t)}{\varphi(t)} < \delta/M$, and choose an ε-cover $\{E_i\}_i$ of E such that $\sum_i \varphi(\mathrm{diam}(E_i)) < M$. Then

$$\begin{aligned} \mathcal{H}^{\psi}_{\varepsilon}(E) &\leq \sum_i \psi(\mathrm{diam}(E_i)) \\ &= \sum_i \varphi(\mathrm{diam}(E_i)) \frac{\psi(\mathrm{diam}(E_i))}{\varphi(\mathrm{diam}(E_i))} < M\frac{\delta}{M} = \delta. \end{aligned}$$

Thus $\mathcal{H}^{\psi}_{\varepsilon} = 0$ and $\mathcal{H}^{\psi}(E) = 0$ as ε may be chosen arbitrarily small.

If $\mathcal{H}^{\varphi}(E)$ is σ-finite, then $E = \cup_i E_i$ where $\mathcal{H}^{\varphi}(E_i) < \infty$ for every i. So by the above $\mathcal{H}^{\psi}(E) \leq \sum_i \mathcal{H}^{\psi}(E_i) = 0$.$\square$

With minor modifications, the above proof shows the following:

- If $\dim(E) \prec \varphi \preceq \psi$ then $\dim(E) \prec \psi$
- If $\dim(E) \prec \varphi \prec \psi$ then $\dim(E) \prec \psi$
- If $\dim(E) \preceq \varphi \preceq \psi$ then $\dim(E) \preceq \psi$

There are reverse inequalities with similar proofs. For example:

Lemma 2.1.2 If $\dim(E) \succeq \varphi \succ \psi$, then $\dim(E) \succ \psi$.

Proof: Let $M > 0$ and choose $\varepsilon > 0$ such that $0 < t < \varepsilon$ implies $\frac{\psi(t)}{\varphi(t)} > M$. Then for any ε-cover $\{U_i\}_i$ of E,

$$\sum_i \psi(\mathrm{diam}(U_i)) = \sum_i \varphi(\mathrm{diam}(U_i)) \frac{\psi(\mathrm{diam}(U_i))}{\varphi(\mathrm{diam}(U_i))} > M\mathcal{H}^{\varphi}_{\varepsilon}(E).$$

Thus for any $M > 0$, $\mathcal{H}^{\psi}(E) \geq M\mathcal{H}^{\varphi}(E)$ so $\mathcal{H}^{\psi}(E) = \infty$.

To see that $\mathcal{H}^{\psi}(E)$ is non-σ-finite suppose that $E = \cup_i E_i$. Then $\mathcal{H}^{\varphi}(E_i) > 0$ for some i and so $\mathcal{H}^{\psi}(E) = \infty$ by the argument above.□

Again a similar proof will show the following:

- If $\dim(E) \succ \varphi \succeq \psi$ then $\dim(E) \succ \psi$
- If $\dim(E) \succ \varphi \succ \psi$ then $\dim(E) \succ \psi$
- If $\dim(E) \succeq \varphi \succeq \psi$ then $\dim(E) \succeq \psi$

Φ is a very rich set and the above ordering is by no means total. It is consequently not a tractable problem to understand how $\dim(E)$ compares with every $\varphi \in \Phi$. What one sometimes does, therefore, is define an appropriate one parameter family,

$\{\varphi_\alpha\}_{\alpha>0} \subset \Phi$, such that $\alpha < \beta$ implies $\varphi_\alpha \prec \varphi_\beta$. Then there is a critical value $\alpha_0 \in [0, \infty]$ such that

$$\mathcal{H}^{\varphi_\alpha}(E) = \begin{cases} 0 & \text{if } \alpha > \alpha_0 \\ \infty & \text{if } \alpha < \alpha_0. \end{cases}$$

Suppose $\psi_\alpha(t) = t^\alpha$. This family of functions is useful when studying subsets of Euclidean space and $\mathcal{H}^{\psi_\alpha}$ is usually abbreviated $\mathcal{H}^\alpha$. I have made the claim that in this dissertation I am primarily interested in infinite dimensional metric spaces. I mean by this more precisely that I will be working with metric spaces (X, ρ) satisfying $\dim(X) \succ \psi_\alpha$ for every $\alpha > 0$. It so happens that a useful family of Hausdorff functions for many of the sets that I will be studying is $\{\psi^\alpha\}_{\alpha>0}$ defined by $\psi^\alpha(t) = 2^{-1/t^\alpha}$. Another useful family is $\{\varphi^\alpha\}_{\alpha>0}$ defined by $\varphi^\alpha(t) = 2^{-\alpha(1/t^s)}$ where $s > 0$ is fixed.

In [Fal2] it is proven that Hausdorff dimension is preserved by bi-Lipschitz transformations. There, however, he is working with the family $\psi_\alpha(t) = t^\alpha$. One needs to be more careful when working with more general functions. The following lemma does hold.

Lemma 2.1.3 Suppose $f : X \to X$ is bi-Lipschitz satisfying:

$$r_1\rho(x, y) \leq \rho(f(x), f(y)) \leq r_2\rho(x, y). \tag{2.1}$$

Then

$$\dim(E) \preceq \varphi \;\Rightarrow\; \dim(f(E)) \preceq \varphi(t/r_2)$$

and

$$\dim(E) \succeq \varphi \;\Rightarrow\; \dim(f(E)) \succeq \varphi(t/r_1).$$

Proof: If $\{E_i\}$ is an ε-cover of E, then $\{f(E_i)\}$ is an $r_2\varepsilon$-cover of $f(E)$. If in addition $\sum_i \varphi(\mathrm{diam}(E_i)) < \infty$, then by the second part of inequality 2.1,

$$\sum_i \varphi\left(\frac{\mathrm{diam}(f(E_i))}{r_2}\right) \leq \sum_i \varphi\left(\frac{r_2 \mathrm{diam}(E_i)}{r_2}\right) < \infty.$$

If $\sum_i \varphi(\mathrm{diam}(E_i)) > c > 0$, then by the first part of inequality 2.1,

$$\sum_i \varphi\left(\frac{\mathrm{diam}(f(E_i))}{r_1}\right) \geq \sum_i \varphi\left(\frac{r_1 \mathrm{diam}(E_i)}{r_1}\right) > c.$$

The result follows.$\square$

Consider, for example, the two parameter family of functions

$$\varphi_{M,\alpha}(t) = 2^{-(M/t)^\alpha}.$$

For a fixed $\alpha > 0$ a bi-Lipschitz map with ratios r_1 and r_2 as above can affect the critical value of M. But it can't be raised by more than a factor $1/r_2$ and it cannot be lowered by more than a factor of $1/r_1$. For $\alpha_1 < \alpha_2$, however, $\varphi_{M_1,\alpha_1} \prec \varphi_{M_2,\alpha_2}$ for any M_1 and M_2. So a bi-Lipschitz map won't have any affect on the critical value of α.

This bi-Lipschitz invariance of the Hausdorff dimension is due to its dependence on the metric structure of the set in question. This is a general feature of fractal dimensions and lemmas similar to 2.1.3 hold for all the notions of dimension discussed here.

2.1.2 The Packing and Centered Covering Dimensions

The packing measure and dimension were introduced by Taylor and Tricot [Tr, TayTr] as notions (almost) dual to the Hausdorff measure and dimension. Equality of the

Hausdorff and packing dimensions of a set imply some regularity properties for the set. The centered covering measure and dimension were introduced by St. Raymond and Tricot [RayTr] as notions precisely dual to the packing measure and dimension. In these papers it is assumed that Hausdorff functions are *blanketed*. That is $\limsup_{t\searrow 0}\varphi(2t)/\varphi(t)<\infty$. This is tantamount to restricting to the finite dimensional case as the following lemma shows.

Lemma 2.1.4 If $\limsup_{t\searrow 0}\varphi(2t)/\varphi(t)<\infty$, then there is an $\alpha>0$ such that $\varphi\prec t^\alpha$.

Proof: Choose $t_0>0$ such that $0<t<t_0$ implies $\varphi(2t)/\varphi(t)<M<\infty$. Then for $n\in\mathbb{N}$,

$$\begin{aligned}\frac{1}{\varphi(2^{-n}t_0)} &= \frac{\varphi(2^{-n+1}t_0)}{\varphi(2^{-n}t_0)}\frac{\varphi(2^{-n+2}t_0)}{\varphi(2^{-n+1}t_0)}\cdots\frac{\varphi(t_0)}{\varphi(2^{-1}t_0)}\frac{1}{\varphi(t_0)}\\ &\le M^n\frac{1}{\varphi(t_0)}.\end{aligned}$$

Thus for $2^{-n}t_0<t\le 2^{-n+1}t_0$,

$$\frac{t^\alpha}{\varphi(t)}\le\frac{(2^{-n+1}t_0)^\alpha}{\varphi(2^{-n}t_0)}\le 2^\alpha\left(M2^{-\alpha}\right)^n\frac{t_0^\alpha}{\varphi(t_0)}.$$

Now as $t\searrow 0$, $n\to\infty$. Thus $\frac{t^\alpha}{\varphi(t)}\to 0$ whenever α is large enough so that $M2^{-\alpha}<1$.$\square$

I'll now define the centered covering and packing measures and investigate the relationships between these quantities and the Hausdorff measure. Of course, I will not assume that my Hausdorff functions are blanketed. This will not affect the definitions, but will slightly alter the comparison theorems.

The *centered covering φ-measure* is very similar to the Hausdorff measure. Rather than covering the set $E\subset X$ with arbitrary sets, however, we cover E with closed balls centered in E. We then measure the size of these balls using their radius rather than their diameter. Thus define a *centered ε-cover* of E to be a finite or countable collection of closed balls $\{B_{r_i}(x_i)\}_i$ such that $x_i \in E$ and $2r_i \le \varepsilon$ for every i. Then let

$$\widetilde{\mathcal{C}}^{\varphi}_{\varepsilon}(E) = \inf\left\{\sum_i \varphi(2r_i) : \{B_{r_i}(x_i)\}_{i=1}^{\infty} \text{ is a centered } \varepsilon\text{-cover of } E\right\},$$

$$\widetilde{\mathcal{C}}^{\varphi}(E) = \lim_{\varepsilon \searrow 0} \widetilde{\mathcal{C}}^{\varphi}_{\varepsilon}(E).$$

As with the Hausdorff measure this is a well-defined limit. However, a centered cover of E may not be a centered cover of some $F \subset E$. Because of this it is possible to have $F \subset E$ with $\widetilde{\mathcal{C}}^{\varphi}(F) > \widetilde{\mathcal{C}}^{\varphi}(E)$. So $\widetilde{\mathcal{C}}^{\varphi}$ is not an outer measure. Therefore define

$$\mathcal{C}^{\varphi}(E) = \sup_{F \subset E} \widetilde{\mathcal{C}}^{\varphi}(F).$$

It is proved in [RayTr] that $\mathcal{C}^{\varphi}$ is a metric outer measure and I will also denote by $\mathcal{C}^{\varphi}$ the restriction of $\mathcal{C}^{\varphi}$ to the $\mathcal{C}^{\varphi}$-measurable sets.

For the *packing φ-measure*, rather than covering E and taking an infimum, one packs E with disjoint closed balls and takes a supremum. More precisely, for $\varepsilon > 0$ an *ε-packing* of E will be a finite or countable collection of disjoint closed balls $\{B_{r_i}(x_i)\}_i$ such that $x_i \in E$ and $2r_i \le \varepsilon$ for every i. Then let

$$\widetilde{\mathcal{P}}^{\varphi}_{\varepsilon}(E) = \sup\left\{\sum_i \varphi(2r_i) : \{B_{r_i}(x_i)\}_i \text{ is an } \varepsilon\text{-packing of } E\right\},$$

$$\widetilde{\mathcal{P}}^{\varphi}(E) = \lim_{\varepsilon \searrow 0} \widetilde{\mathcal{P}}^{\varphi}_{\varepsilon}(E).$$

This time the limit exists because as ε decreases $\widetilde{\mathcal{P}}^{\varphi}_{\varepsilon}(E)$ decreases as well.

We shall see shortly that $\widetilde{\mathcal{P}}^\varphi$ is not sub-σ-additive and, so, not an outer measure. So, for σ-totally bounded $E \subset X$ define

$$\mathcal{P}^\varphi(E) = \inf\left\{\sum_i \widetilde{\mathcal{P}}^\varphi(E_i) : E \subset \cup_i E_i\right\}.$$

Note that although $\widetilde{\mathcal{P}}^\varphi$ is not an outer measure, it is finitely subadditive and monotone. That is if $E \subset \cup_{i=1}^n E_i$, then $\widetilde{\mathcal{P}}^\varphi(E) \leq \sum_{i=1}^n \widetilde{\mathcal{P}}^\varphi(E_i)$ and if $F \subset E$, then $\widetilde{\mathcal{P}}^\varphi(F) \leq \widetilde{\mathcal{P}}^\varphi(E)$. Both these statements follow immediately from the fact that a packing of a subset of E is also a packing of E. Also note that $\mathcal{P}^\varphi$ is finite for a broader class of sets than $\widetilde{\mathcal{P}}^\varphi$. If E is σ-totally bounded, but not totally bounded then $\widetilde{\mathcal{P}}^\varphi(E) = \infty$ for any $\varphi \in \Phi$. Next I will show that $\widetilde{\mathcal{P}}^\varphi$ respects closure. That is $\widetilde{\mathcal{P}}^\varphi(E) = \widetilde{\mathcal{P}}^\varphi(\overline{E})$. Thus we may assume that the E_i's above are closed sets.

Lemma 2.1.5 For all totally bounded sets E, $\widetilde{\mathcal{P}}^\varphi(E) = \widetilde{\mathcal{P}}^\varphi(\overline{E})$.

Proof: If $\{B_{r_i}(x_i)\}_i$ is an ε-packing of $\overline{E}$, then we may choose an ε-packing $\{B_{r_i'}(x_i')\}_i$ of E with $\rho(x_i, x_i')$ and $r_i - r_i' > 0$ as small as we like. Thus $\sum_i \varphi(2r_i) - \sum_i \varphi(2r_i') > 0$ may be made as small as we like. So $\widetilde{\mathcal{P}}^\varphi(E) = \widetilde{\mathcal{P}}^\varphi(\overline{E})$.□

The following example illustrates how the lack of sub-σ-additivity of $\widetilde{\mathcal{P}}^\varphi$ extends into the infinite dimensional realm.

Example 2.1.1 Let $\varphi \in \Phi$ and $a_n \in \varphi^{-1}(\{1/n\})$. Note that $a_n \searrow 0$. Let $X = \{x_0, x_1, x_2, \ldots, x_\infty\}$ be a countable set. Define a metric ρ on X by

$$\rho(x_n, x_m) = \begin{cases} a_n & \text{if } n \neq m = \infty \\ a_n + a_m & \text{if } \infty \neq n \neq m \neq \infty \\ 0 & \text{if } n = m. \end{cases}$$

Of course, X is a countable union of singletons each satisfying $\widetilde{\mathcal{P}}^{\varphi}(\{x_n\}) = 0$. However, if $a_{n+1} < \varepsilon \leq a_n$, then considering the packing with $n+1$ balls of radius $\varepsilon/2$ centered at $x_0, x_1, \ldots, x_n$ we get

$$\widetilde{\mathcal{P}}^{\varphi}_{\varepsilon}(X) \geq (n+1)\varphi\left(2\frac{\varepsilon}{2}\right) \geq \frac{n+1}{n+1} = 1.$$

Thus $\widetilde{\mathcal{P}}^{\varphi}(X) \geq 1$.

All the quantities introduced so far reflect the asymptotic properties of $\varphi(t)$ as $t \searrow 0$ in a way similar to lemma 2.1.1. There is also the bi-Lipschitz equivalence as in lemma 2.1.3. The proofs are all similar. There are also relationships between these quantities as the following theorem states.

Theorem 2.1.1 Suppose that $\tau, \varphi, \psi \in \Phi$ satisfy

$$\limsup_{t\searrow 0} \frac{\tau(2t)}{\varphi(t)} < A < \infty, \quad \limsup_{t\searrow 0} \frac{\varphi(2t)}{\psi(t)} < M < \infty. \tag{2.2}$$

Then for $E \subset X$,

$$\frac{1}{A}\mathcal{C}^{\tau}(E) \leq \mathcal{H}^{\varphi}(E) \leq \widetilde{\mathcal{C}}^{\varphi}(E) \leq \mathcal{C}^{\varphi}(E) \leq M\mathcal{P}^{\psi}(E) \leq M\widetilde{\mathcal{P}}^{\psi}(E).$$

Proof: I first prove $\frac{1}{A}\widetilde{\mathcal{C}}^{\tau}(F) \leq \mathcal{H}^{\varphi}(F)$ for every $F \subset E$. Let $F \subset E$, let $\varepsilon > 0$ be small enough so that $0 < t \leq \varepsilon$ implies $\tau(2t) \leq A\varphi(t)$, and choose an ε-cover $\{E_i\}_i$ of F. We may assume that $E_i \cap F \neq \emptyset$ for all i. Choose for every i an $x_i \in E_i \cap F$. Then $\{B_{\mathrm{diam}(E_i)}(x_i)\}_i$ forms a centered 2ε-cover of F. Now by the first inequality in 2.2,

$$\sum_i \tau(2\mathrm{diam}(E_i)) \leq A \sum_i \varphi(\mathrm{diam}(E_i)).$$

So $\widetilde{\mathcal{C}}^{\tau}_{2\varepsilon}(F) \le A\mathcal{H}^{\varphi}_{\varepsilon}(F)$ and $\frac{1}{A}\widetilde{\mathcal{C}}^{\tau}(F) \le \mathcal{H}^{\varphi}(F)$. Now $\mathcal{H}^{\varphi}$ is monotone so

$$\frac{1}{A}\mathcal{C}^{\tau}(E) = \frac{1}{A}\sup_{F\subset E}\widetilde{\mathcal{C}}^{\tau}(F) \le \sup_{F\subset E}\mathcal{H}^{\varphi}(F) = \mathcal{H}^{\varphi}(E).$$

$\mathcal{H}^{\varphi}(E) \le \widetilde{\mathcal{C}}^{\varphi}(E) \le \mathcal{C}^{\varphi}(E)$ is immediate. For the remaining inequalities, it suffices to prove that $\widetilde{\mathcal{C}}^{\varphi}(E) \le M\widetilde{\mathcal{P}}^{\psi}(E)$. If this is the case then,by the monotonicity of $\widetilde{\mathcal{P}}^{\varphi}$, for $F \subset E$,

$$\widetilde{\mathcal{C}}^{\varphi}(F) \le M\widetilde{\mathcal{P}}^{\psi}(F) \le M\widetilde{\mathcal{P}}^{\psi}(E).$$

Thus $\mathcal{C}^{\varphi}(E) \le M\widetilde{\mathcal{P}}^{\psi}(E)$ upon taking supremums. Next, if $E = \cup_i E_i$, then

$$\mathcal{C}^{\varphi}(E) \le \sum_i \mathcal{C}^{\varphi}(E_i) \le M\sum_i \widetilde{\mathcal{P}}^{\psi}(E_i).$$

Thus $\mathcal{C}^{\varphi}(E) \le M\mathcal{P}^{\psi}(E)$ upon taking infimums.

So finally I show that $\widetilde{\mathcal{C}}^{\varphi}(E) \le M\widetilde{\mathcal{P}}^{\psi}(E)$. We may assume that $\widetilde{\mathcal{P}}^{\psi}(E) < \infty$ which implies that E is totally bounded. By the second inequality of 2.2, we may choose $\varepsilon > 0$ small enough so that $0 < t \le \varepsilon$ implies $\varphi(2t) < M\psi(t)$ and let $\{B_{\varepsilon/2}(x_i)\}_{i=1}^n$ be an ε-packing of E maximal in the sense that for any $x \in E$, $B_{\varepsilon/2}(x) \cap B_{\varepsilon/2}(x_i) \ne \emptyset$ for some i. Then $\{B_{\varepsilon}(x_i)\}_{i=1}^n$ is a 2ε-cover of E and

$$\sum_{i=1}^n \varphi(2\varepsilon) \le M\sum_{i=1}^n \psi(\varepsilon).$$

So $\widetilde{\mathcal{C}}^{\varphi}_{2\varepsilon}(E) \le M\widetilde{\mathcal{P}}^{\psi}_{\varepsilon}(E)$ and $\widetilde{\mathcal{C}}^{\varphi}(E) \le M\widetilde{\mathcal{P}}^{\psi}(E)$ upon letting $\varepsilon \searrow 0$.$\square$

2.2 The Entropy Dimensions

In this sections I define the entropy dimensions in terms of the earlier notion of an entropy index. The entropy indices seem to have been introduced by a number of different authors. The earliest references I am aware of are [Bou] and [PonSc]. They were generalized and studied in function space in [KolTi]. The modification to obtain the entropy dimensions goes back at least to [Weg].

The basic idea for the entropy indices is to introduce for a totally bounded subset $E \subset X$ and $\varepsilon > 0$ a quantity $N_\varepsilon(E)$ which behaves asymptotically as $\varepsilon \searrow 0$ in a way indicative of the dimension of E. Here define

$$N_\varepsilon(E) = \text{max number of closed disjoint } \frac{\varepsilon}{2}\text{-balls centered in } E,$$

where $\varepsilon/2$ refers to the radius of the ball. Then, if I^n is an n-dimensional cube, $N_\varepsilon(I^n) \asymp (1/\varepsilon)^n$ ([KolTi] sec.4). I will be interested in comparing $N_\varepsilon(E)$ with more general Hausdorff functions. So for $\varphi \in \Phi$ write:

- $\Delta(E) \prec \varphi$ if $\limsup_{\varepsilon \searrow 0} \varphi(\varepsilon) N_\varepsilon(E) = 0$
- $\Delta(E) \succ \varphi$ if $\limsup_{\varepsilon \searrow 0} \varphi(\varepsilon) N_\varepsilon(E) = \infty$.
- $\Delta(E) \preceq \varphi$ if $\limsup_{\varepsilon \searrow 0} \varphi(\varepsilon) N_\varepsilon(E) < \infty$.
- $\Delta(E) \succeq \varphi$ if $\limsup_{\varepsilon \searrow 0} \varphi(\varepsilon) N_\varepsilon(E) > 0$.
- $\Delta(E) \asymp \varphi$ if $0 < \limsup_{\varepsilon \searrow 0} \varphi(\varepsilon) N_\varepsilon(E) < \infty$.

This defines the *upper entropy index*, Δ, through comparison of N_ε with Hausdorff functions using the $\limsup$. The lower entropy index, δ, is similar but we use the $\liminf$. Thus:

- $\delta(E) \prec \varphi$ if $\liminf_{\varepsilon \searrow 0} \varphi(\varepsilon) N_\varepsilon(E) = 0$

- $\delta(E) \succ \varphi$ if $\liminf_{\varepsilon \searrow 0} \varphi(\varepsilon) N_\varepsilon(E) = \infty$.

and similarly for $\preceq$, $\succeq$, and $\asymp$.

As we shall see, Δ is closely related to $\widetilde{\mathcal{P}}^\varphi$ and suffers from defects related to $\widetilde{\mathcal{P}}^\varphi$'s lack of sub-$\sigma$-additivity. We therefore modify the definition to obtain the *upper entropy dimension* $\widehat{\Delta}$ as follows. By a *decomposition* of E, I mean a finite or countable collection of sets $\{E_i\}_i$ whose union is E. Write:

- $\widehat{\Delta}(E) \prec \varphi$ (resp. $\widehat{\Delta}(E) \preceq \varphi$) if there is a decomposition $E = \cup_i E_i$ so that $\Delta(E_i) \prec \varphi$ (resp. $\Delta(E_i) \preceq \varphi$) for every i

- $\widehat{\Delta}(E) \succ \varphi$ (resp. $\widehat{\Delta}(E) \succeq \varphi$) if for every decomposition $E = \cup_i E_i$ there is an i so that $\Delta(E_i) \succ \varphi$ (resp.$\Delta(E_i) \succeq \varphi$).

Similar definitions may be made to define $\widehat{\delta}$ in terms of δ.

Some comments on these definitions are in order:

1. Since a packing by $\varepsilon/2$-balls of a subset of E is also one of E, Δ, $\widehat{\Delta}$, δ, and $\widehat{\delta}$ are all monotone. That is if $F \subset E$, then $\Delta(E) \prec \varphi$ implies $\Delta(F) \prec \varphi$ and $\Delta(E) \preceq \varphi$ implies $\Delta(F) \preceq \varphi$.

2. Although Δ and δ are defined only for totally bounded sets, $\widehat{\Delta}$ and $\widehat{\delta}$ are defined for σ-totally bounded sets.

3. Δ and δ respect closure. The proof is similar to lemma 2.1.5 We may therefore assume that the E_i's in the definition of $\widehat{\Delta}$ and $\widehat{\delta}$ are closed.

The concept of *σ-stability* is a property of $\widehat{\Delta}$ and $\widehat{\delta}$ defined by the following lemma.

Lemma 2.2.1 Suppose that $E = \cup_n E_n$. Then $\widehat{\Delta}(E_n) \prec \varphi$ for every n implies $\widehat{\Delta}(E) \prec \varphi$. $\widehat{\Delta}(E) \succ \varphi$ implies there is an n such that $\widehat{\Delta}(E_n) \succ \varphi$. Similar statements hold with $\widehat{\delta}$, $\succeq$, or $\preceq$ replacing $\widehat{\Delta}$, $\succ$, or $\prec$ respectively.

Proof: For the first part, if $\widehat{\Delta}(E_n) \prec \varphi$ for every n, then each E_n may be decomposed $E_n = \cup_i E_{n,i}$ where $\Delta(E_{n,i}) \prec \varphi$ for every i. Then $E = \cup_n \cup_i E_{n,i}$ is a decomposition of E implying $\widehat{\Delta}(E) \prec \varphi$.

For the second part, suppose that each E_n is further decomposed $E_n = \cup_k E_{n,k}$. Then $E = \cup_n \cup_k E_{n,k}$. Since $\widehat{\Delta}(E) \succ \varphi$, there are n and k such that $\Delta(E_{n,k}) \succ \varphi$. $E_{n,k}$ is part of an arbitrary decomposition of E_n. This says that this n satisfies $\widehat{\Delta}(E_n) \succ \varphi$. The other proofs are similar.$\square$

Example 2.2.1 Let $\varphi \in \Phi$ and let (X, ρ) be the corresponding metric space as in example 2.1.1. Then the same argument used there shows that $\Delta(X) \succ \varphi$ and even that $\delta(X) \succ \varphi$. So δ and Δ are not σ-stable as X is countable.

Next I investigate the relationship between Δ and $\widetilde{\mathcal{P}}^{\varphi}$. Let $\varphi \in \Phi$. A *blanketing sequence* for φ is a sequence $a_n \searrow 0$ such that $\varphi(a_n) \le 2\varphi(a_{n+1})$. Given $\varphi \in \Phi$ such a sequence may always be chosen recursively.

Theorem 2.2.1 Let $E \subset X$ be totally bounded. Then $\widetilde{\mathcal{P}}^{\varphi}(E) < \infty$ implies $\Delta(E) \preceq \varphi$. Conversely, suppose that $(a_n)_{n=1}^{\infty}$ is a blanketing sequence for φ and

$$\sum_{n=1}^{\infty} \frac{\psi(a_n)}{\varphi(a_n)} < \infty. \tag{2.3}$$

Then $\Delta(E) \preceq \varphi$ implies $\widetilde{\mathcal{P}}^{\psi}(E) = 0$.

Proof: For the first statement note that for $\varepsilon > 0$ we have $N_\varepsilon(E)\varphi(\varepsilon) \leq \widetilde{\mathcal{P}}^\varphi_\varepsilon(E)$ since the supremum on the right is taken over more possible packings than the one on the left. So if $\widetilde{\mathcal{P}}^\varphi_\varepsilon(E) \to \widetilde{\mathcal{P}}^\varphi(E) < \infty$, then $\Delta(E) \preceq \varphi$.

For the second statement let $\delta > 0$ and choose $\varepsilon_0, M > 0$ such that $0 < \varepsilon \leq \varepsilon_0$ implies $N_\varepsilon(E)\varphi(\varepsilon) < M$. Choose $n_0 \in \mathbb{N}$ large enough so that $a_{n_0} \leq \varepsilon_0$ and

$$\sum_{n=n_0}^{\infty} \frac{\psi(a_n)}{\varphi(a_n)} < \frac{\delta}{2M}.$$

Let $\varepsilon \in (0, a_{n_0}]$ and choose an ε-packing $\mathcal{B}$ of E. For $n \geq n_0$ let $\mathcal{B}_n = \{B_r(x) \in \mathcal{B} : a_{n+1} < 2r \leq a_n\}$. Note that $\#(\mathcal{B}_n) \leq N_{a_{n+1}}(E)$. So by these choices and inequality 2.3,

$$\begin{aligned}
\sum_{B_r(x)\in\mathcal{B}} \psi(2r) &= \sum_{n=n_0}^{\infty} \sum_{B_r(x)\in\mathcal{B}_n} \psi(2r) \leq \sum_{n=n_0}^{\infty} N_{a_{n+1}}(E)\psi(a_n) \\
&= \sum_{n=n_0}^{\infty} N_{a_{n+1}}(E)\varphi(a_{n+1}) \frac{\psi(a_n)}{\varphi(a_n)} \frac{\varphi(a_n)}{\varphi(a_{n+1})} \\
&\leq 2M \sum_{n=n_0}^{\infty} \frac{\psi(a_n)}{\varphi(a_n)} < 2M \frac{\delta}{2M} = \delta.
\end{aligned}$$

Thus $\widetilde{\mathcal{P}}^\psi_\varepsilon(E) = 0$ as $\delta > 0$ was arbitrary. Since this is true for every $\varepsilon > 0$ we have $\widetilde{\mathcal{P}}^\psi(E) = 0$.□

This theorem says that the upper entropy index and the packing premeasure lead to almost the same notion of dimension. When working with a family of Hausdorff functions $(\psi_s)_{s>0}$ the packing premeasure and upper entropy index will frequently lead to the same critical value. For example if $\psi_s(\varepsilon) = \varepsilon^s$, then $a_n = 2^{-(n/s)}$ defines a blanketing sequence for ψ_s. If $s < t$, then

$$\sum_n \frac{\psi_t(a_n)}{\psi_s(a_n)} = \sum_n 2^{n(1-(t/s))} < \infty.$$

And so the packing premeasure and upper entropy index lead to the same critical value for the family $(\psi_s)_{s>0}$. More precisely

$$\sup\{s>0 : \widetilde{\mathcal{P}}^{\psi_s}(E)=\infty\} = \sup\{s>0 : \Delta(E)\succ\psi_s\}.$$

As another example, suppose $\psi^\alpha(\varepsilon)=2^{-\alpha(1/\varepsilon)^s}$. Then $a_n=(\frac{\alpha}{n})^{1/s}$ is a blanketing sequence for $(\psi^\alpha)_{\alpha>0}$. If $\alpha<\beta$, then

$$\sum_n \frac{\psi^\beta(a_n)}{\psi^\alpha(a_n)} = \sum_n \frac{2^{-\beta((n/\alpha)^{1/s})^s}}{2^{-\alpha((n/\alpha)^{1/s})^s}} = \sum_n 2^{n(1-(\beta/\alpha))} < \infty.$$

So again the upper entropy index and packing premeasure lead to the same critical value for the family $(\psi^\alpha)_{\alpha>0}$.

More important is the close relationship between the upper entropy dimension and the packing dimension stated in the following theorem.

Theorem 2.2.2 Let $E\subset X$ be σ-totally bounded. If $\mathcal{P}^\varphi(E)$ is σ-finite and $\varphi\preceq\psi$, then $\widehat{\Delta}(E)\preceq\psi$. Conversely, if $(a_n)_n$ is a blanketing sequence for φ and

$$\sum_{n=1}^{\infty}\frac{\psi(a_n)}{\varphi(a_n)}<\infty,$$

then $\widehat{\Delta}(E)\preceq\varphi$ implies $\mathcal{P}^\psi(E)=0$.

Proof: For the first part, assume first that $\mathcal{P}^\varphi(E)<M<\infty$. Then there is a decomposition $E=\cup_i E_i$ where $\sum_i\widetilde{\mathcal{P}}^\varphi(E_i)<M$. So $\widetilde{\mathcal{P}}^\varphi(E_i)<M$ for every i. Since $N_\varepsilon(E_i)\varphi(\varepsilon)\le\widetilde{\mathcal{P}}^\varphi_\varepsilon(E_i)$, we have

$$\limsup_{\varepsilon\searrow 0} N_\varepsilon(E_i)\varphi(\varepsilon) \le \lim_{\varepsilon\searrow 0}\widetilde{\mathcal{P}}^\varphi_\varepsilon(E_i) < M.$$

So

$$\begin{aligned}\limsup_{\varepsilon \searrow 0} N_\varepsilon(E_i)\psi(\varepsilon) &= \limsup_{\varepsilon \searrow 0} N_\varepsilon(E)\varphi(\varepsilon)\frac{\psi(\varepsilon)}{\varphi(\varepsilon)} \\ &\leq M \limsup_{\varepsilon \searrow 0} \frac{\psi(\varepsilon)}{\varphi(\varepsilon)} < \infty.\end{aligned}$$

So $\widehat{\Delta}(E) \preceq \psi$. If $\mathcal{P}^\varphi(E)$ is σ-finite, then $E = \cup_n E_n$ where $\mathcal{P}^\varphi(E_n) < \infty$ for every n. So $\widehat{\Delta}(E_n) \preceq \psi$ for every n by the above and $\widehat{\Delta}(E) \preceq \psi$ by σ-stability (lemma 2.2.1).

For the second part, if $\widehat{\Delta}(E) \preceq \varphi$, then there is a decomposition $E = \cup_i E_i$ such that $\Delta(E_i) \preceq \varphi$ for every i. So $\widetilde{\mathcal{P}}^\psi(E_i) = 0$ by theorem 2.2.1 and

$$\mathcal{P}^\psi(E) \leq \sum_{i=1}^{\infty} \widetilde{\mathcal{P}}^\psi(E_i) = 0.\Box$$

CHAPTER III

Computational Methods

The definitions of the preceding chapter are useful theoretically, but cumbersome computationally. In this chapter, I'll develop some computational tools and illustrate them with some simple examples. As shown in theorems 2.1.1, 2.2.1, and 2.2.2, many definitions of dimension are essentially equivalent to others. From this point on, I will concentrate on the Hausdorff and upper entropy dimensions. Conclusions about other dimensions may be drawn using the appropriate comparison theorem 2.1.1, 2.2.1, or 2.2.2.

3.1 Sequence Spaces

I begin by introducing a family of compact, totally disconnected metric spaces which are relatively easy to deal with and will aid in later analysis. For $k \in \mathbb{N}^+$ let $a_k \in \mathbb{N}^+$, let$A_k = \{1, \ldots, a_k\}$ be a discrete set, and let $\Omega = \prod_{j=1}^{\infty} A_j$. Thus $\sigma = (\sigma_j)_{j=1}^{\infty} \in \Omega$ if $\sigma_j \in A_j$ for every j. I will define a metric d on Ω inducing the product topology on Ω. First I develop some useful notation. Given $n \in \mathbb{N}$ an *initial segment* of length n is a finite sequence $\alpha = (\alpha_j)_{j=1}^{n}$ with $\alpha_j \in A_j$ for every $j = 1, \ldots, n$. There is by definition one initial segment of length zero namely the empty segment denoted Λ. If α is an initial segment, write $|\alpha|$ to denote the length of α. For $n \in \mathbb{N}$, let Ω^n denote the set of all initial segments of length n. Let $\Omega^* = \cup_{j=0}^{\infty} \Omega^j$ be the set of all initial segments. If $\sigma \in \Omega$ write $\sigma|_n$ for the initial segment $(\sigma_1, \ldots, \sigma_n) \in \Omega^n$. We may put a partial order on Ω^* as follows: For $\alpha, \beta \in \Omega^*$, say $\alpha = (\alpha_1, \ldots, \alpha_j)$ and $\beta = (\beta_1, \ldots, \beta_k)$, write $\alpha < \beta$ if $j < k$ and $\alpha_i = \beta_i$ for $i = 1, \ldots, j$. If $\alpha < \beta$, then β is said to be a descendant of α. If $\alpha \in \Omega^n$, then let α^- denote the unique element of Ω^{n-1} such that $\alpha^- < \alpha$. α^- is called the *parent* and α the *child*. Also if $\alpha \in \Omega^n$ is an initial segment, then let

$$[\alpha] = \{\sigma \in \Omega : \sigma_i = \alpha_i \text{ for } i = 1, \ldots, n\}.$$

To define a metric d on Ω associate to each $\alpha \in \Omega^*$ a number $r(\alpha) > 0$ satisfying the following:

1. $r(\Lambda) > 0$,

2. $r(\beta) < r(\alpha)$ whenever $\beta > \alpha$,

3. $r(\sigma|_n) \to 0$ as $n \to \infty$.

If $\sigma, \tau \in \Omega$ have α as their longest common initial segment then define $d(\sigma, \tau) = r(\alpha)$. Define, also, $d(\sigma, \sigma) = 0$. Then d is a metric inducing the product topology on Ω. For details see [Ed1]. Note that if $\sigma, \tau \in \Omega$ and $r(\sigma|_n) \leq \varepsilon < r(\sigma|_{n-1})$, then $\rho(\sigma, \tau) \leq \varepsilon$ is equivalent to $\sigma_i = \tau_i$ for $i = 1, \ldots, n$. Thus $B_\varepsilon(\sigma) = [\sigma|_n]$.

There are two specific types of sequence spaces which will be of particular interest here. For the first, let $s > 0$ and let $A_k = \{1, \ldots, 2^{2^{k-1}}\}$. Choose $r \in (0, 1)$ such that

$1/r^s = 2$. If $\alpha \in \Omega^n$, define $r(\alpha) = r^n$. The resulting sequence space will be called *infinite s-space*. We will see in the following sections that an appropriate Hausdorff function to describe the dimension of this space is given by $\psi^s(\varepsilon) = 2^{-(1/\varepsilon)^s}$.

The other case of interest has been used in the analysis of self-similar sets (see [Ed1]). Let $m \in \mathbb{N}$ be fixed and let $A_k = \{1, \dots, m\}$ for every k. Let $(r_1, \dots, r_m)$ satisfy $0 < r_i < 1$ for every $i = 1, \dots, m$. Such a list is called a *contraction ratio list*. For $\alpha \in \Omega^n$, let $r(\alpha) = \prod_{i=1}^n r_{\alpha_i}$. In [Ed1] it is shown that the dimension of this sequence space is given by $\psi_s(\varepsilon) = \varepsilon^s$, where $\sum_{i=1}^m r_i^s = 1$. In fact it is shown that

$$\mathcal{H}^s([\alpha]) = r(\alpha)^s \tag{3.1}$$

for every $\alpha \in \Omega^*$. This space is called *self-similar sequence space* and I will be using it when studying hyperspaces.

3.2 Calculation of the Entropy Dimensions

In this section I will calculate the entropy dimensions for infinite s-space. To estimate the entropy indices of a totally bounded set E it is necessary to obtain upper and lower bounds for $N_\varepsilon(E)$. As N_ε is defined in terms of a supremum, it is relatively easy to find a lower bound, but frequently difficult to find an upper bound. The special nature of sequence space alleviates this problem somewhat.

Theorem 3.2.1 Fix $s > 0$, choose $r \in (0, 1)$ so that $r^{-s} = 2$, and let (Ω, ρ) be the corresponding infinite s-space. Then $\Delta(\Omega) \asymp \psi^s(r\varepsilon/2)$, where $\psi^s(\varepsilon) = 2^{-(1/\varepsilon)^s}$.

Proof: Suppose $2r^k \le \varepsilon < 2r^{k-1}$. I claim then that $N_\varepsilon(\Omega) = 2^{2^k-1}$. This is because every ball of radius $\varepsilon/2 \in [r^k, r^{k-1})$ is of the form $[\alpha]$ for some $\alpha \in \Omega^k$. Furthermore if $\alpha, \beta \in \Omega^k$ then $[\alpha] \cap [\beta] = \emptyset$. So

$$\begin{aligned} N_\varepsilon(\Omega) &= \#(\Omega^k) = \prod_{i=1}^{k} \#(A_k) = \prod_{i=1}^{k} 2^{2^{k-1}} \\ &= 2^{2^{k-1}+2^{k-2}+\cdots+1} = 2^{2^k-1}. \end{aligned}$$

So

$$N_\varepsilon(\Omega)\psi^s(r\varepsilon/2) = 2^{2^k-1}2^{-(2/r\varepsilon)^s} \le 2^{2^k-1}2^{-(2/2r^k)^s} = 2^{2^k-1}2^{-2^k} = \frac{1}{2}.$$

Thus $\limsup_{\varepsilon\to 0} N_\varepsilon(\Omega)\psi^s(r\varepsilon/2) \le \frac{1}{2}$ and $\Delta(\Omega) \preceq \psi^s(r\varepsilon/2)$.

For the lower bound, choose for every $k \in \mathbb{N}$ an $\varepsilon_k \in [2r^k, 2r^{k-1})$ so that $\psi^s(r\varepsilon_k/2) \ge \frac{1}{2}\psi^s(r^k)$. Then

$$N_{\varepsilon_k}(\Omega)\psi^s(r\varepsilon_k/2) \ge 2^{2^k-1}2^{-(1/r^k)^s-1} = \frac{1}{4}$$

and

$$\limsup_{\varepsilon\to 0} N_\varepsilon(\Omega)\psi^s(r\varepsilon/2) \ge \limsup_{k\to\infty} N_{\varepsilon_k}(\Omega)\psi^s(r\varepsilon_k/2) \ge \frac{1}{4}.$$

Thus $\Delta(\Omega) \succeq \psi^s(r\varepsilon/2)$ and $\Delta(\Omega) \asymp \psi^s(r\varepsilon/2)\Box$.

A similar proof shows that $\delta(\Omega) \asymp \psi^s(\varepsilon/2)$. Note that $\psi^s(\varepsilon/2) \prec \psi^s(r\varepsilon/2)$.

Next, we need to pass from the entropy index to the entropy dimension. Clearly if $\Delta(E) \preceq \varphi$, then $\widehat{\Delta}(E) \preceq \varphi$, since (E) is a decomposition of E. A similar statement holds with $\preceq$ replaced by $\prec$. The key to getting lower bounds on $\widehat{\Delta}$ is in the next lemma.

Lemma 3.2.1 Suppose that E is a closed subset of the complete metric space X and $\Delta(E \cap U) \succeq \varphi$ for every open set $U \subset X$. Then $\widehat{\Delta}(E) \succeq \varphi$. A similar statement holds with $\succeq$ replaced by $\succ$.

Proof: Let $(E_i)_i$ be a decomposition of E by closed sets. As E is a closed subset of a complete metric space, we may apply the Baire category theorem to obtain an E_i which is somewhere dense in E. This means there is an open set $U \subset X$ such that $E \cap U \subset E_i$. So $\Delta(E_i) \succeq \varphi$ by the monotonicity of Δ and $\widehat{\Delta}(E) \succeq \varphi$. The same proof works for $\succ$.$\Box$

Theorem 3.2.2 If (Ω, ρ) is infinite s-space, then $\widehat{\Delta}(\Omega) \succeq \psi^s(r\varepsilon/2)$.

Proof: First I'll prove that $\Delta([\alpha]) \succeq \psi^s(r\varepsilon/2)$ for every $\alpha \in \Omega^*$. This implies the result when combined with lemma 3.2.1. Fix $n \in \mathbb{N}$ and suppose that $\alpha \in \Omega^n$. If $k > n$ and $2r^k \leq \varepsilon < 2r^{k-1}$, then considerations similar to those when calculating $N_\varepsilon(\Omega)$ show that

$$\begin{aligned} N_\varepsilon([\alpha]) &= \prod_{i=n+1}^{k} \#(A_i) = \prod_{i=n+1}^{k} 2^{2^{i-1}} \\ &= 2^{2^{k-1}+2^{k-2}+\cdots+2^n} = 2^{2^k-2^n}. \end{aligned}$$

Now for every $k \in \mathbb{N}$ choose $\varepsilon_k \in [2r^k, 2r^{k-1})$ such that $\psi^s(r\varepsilon_k/2) \geq \frac{1}{2}\psi^s(r^k)$. Then,

$$\begin{aligned} N_{\varepsilon_k}([\alpha])\psi^s(r\varepsilon_k/2) &\geq 2^{2^k-2^n}\left(\frac{1}{2}\psi^s(r^k)\right) \\ &= \frac{1}{2}2^{2^k-2^n}2^{-2^k} = 2^{2^k-2^n-2^k-1} = 2^{-(2^n+1)}. \end{aligned}$$

So

$$\limsup_{\varepsilon\to 0} N_\varepsilon([\alpha])\psi^s(r\varepsilon/2) \geq \limsup_{k\to\infty} N_{\varepsilon_k}([\alpha])\psi^s(r\varepsilon_k/2) > 2^{-(2^n+1)}.$$

Thus $\Delta([\alpha]) \succeq \psi^s(r\varepsilon/2)$ and $\widehat{\Delta}(\Omega) \asymp \psi^s(r\varepsilon/2)$.$\square$

A similar argument shows that $\widehat{\delta}(\Omega) \asymp \psi^s(\varepsilon/2)$.

3.3 Computation of Hausdorff Measures

Now I turn to the computation of Hausdorff measures. In this section $\psi^s(t) = 2^{-(1/t)^s}$. As Hausdorff measure is defined in terms of infimums, upper bounds are relatively straight forward.

Lemma 3.3.1 If (Ω, ρ) is infinite s-space, then $\mathcal{H}^{\psi^s}(\Omega) \leq 1/2$.

Proof: Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ such that $r^k < \varepsilon$ where $1/r^s = 2$. Then $\{[\alpha] : \alpha \in \Omega^k\}$ is an ε-cover of Ω with 2^{2^k-1} sets of diameter r^k. So

$$\mathcal{H}^{\psi^s}_{\varepsilon}(\Omega) \leq 2^{2^k-1}\psi^s(r^k) = 2^{2^k-1}2^{-(1/r^k)^s} = 2^{2^k-1}2^{-2^k} = \frac{1}{2}.$$

Thus $\mathcal{H}^{\psi^s}(\Omega) = \lim_{\varepsilon\to 0} \mathcal{H}^{\psi^s}_{\varepsilon}(\Omega) \leq \frac{1}{2}$.$\square$

Lower bounds may be obtained by comparing $\mathcal{H}^{\varphi}$ with other measures as in the following lemma.

Lemma 3.3.2 Let $\varphi \in \Phi$ and suppose there exists $c, \delta > 0$ and a positive Borel measure μ on E such that $\mu(U) < c\varphi(\mathrm{diam}(U))$ for every set $U \subset X$ with $\mathrm{diam}(U) \leq \delta$. Then $\mu(E) \leq c\mathcal{H}^{\varphi}(E)$.

Proof: Suppose $0 < \varepsilon \le \delta$ and $(U_i)_i$ is an ε-cover of E. Then

$$0 < \mu(E) \le \sum_i \mu(U_i) \le c \sum_i \varphi(\mathrm{diam}(U_i)).$$

So $\mu(E) \le c\mathcal{H}^\varphi_\varepsilon(E) \le c\mathcal{H}^\varphi(E)$ after taking infimums.□

Next, I apply this lemma to infinite s-space, (Ω, ρ). First some notation: If $\alpha \in \Omega^k$ and $\theta \in A_{k+1}$, let $\alpha\theta \in \Omega^{k+1}$ denote the concatenation of α with θ. Thus $\alpha\theta = (\alpha_1, \ldots, \alpha_k, \theta)$. Now for $\alpha \in \Omega^k$ define $\widetilde{\mu}([\alpha]) = 2^{-2^k}$. To extend $\widetilde{\mu}$ to finite unions of such sets, note that if $\alpha \in \Omega^k$ then

$$\sum_{\theta \in A_{k+1}} \widetilde{\mu}([\alpha\theta]) = 2^{2^k} 2^{-2^{k+1}} = 2^{-2^k} = \widetilde{\mu}([\alpha]).$$

It follows by induction that if $\mathcal{F} \in \Omega^*$ is finite and $[\alpha] \cap [\beta] = \emptyset$, for every $\alpha, \beta \in \mathcal{F}$ with $\alpha \neq \beta$, then

$$\widetilde{\mu}(\cup_{\alpha \in \mathcal{F}} [\alpha]) = \sum_{\alpha \in \mathcal{F}} \widetilde{\mu}([\alpha])$$

defines $\widetilde{\mu}$ consistently on the algebra of sets $\mathcal{A}$ generated by $\{[\alpha] : \alpha \in \Omega^*\}$. We may use $\widetilde{\mu}$ to define an outer measure μ^* by using what is sometimes called *method I* (see [Ed1])

$$\mu^*(E) = \inf \left\{ \sum_{A \in \mathcal{C}} \widetilde{\mu}(A) : E \subset \cup_{A \in \mathcal{C}} A \text{ and } \mathcal{C} \subset \mathcal{A} \right\}.$$

Finally, let μ denote the restriction of μ^* to the μ^*-measurable subsets of Ω. Since $\widetilde{\mu}$ is finitely additive on disjoint subsets of $\mathcal{A}$, it follows that each $A \in \mathcal{A}$ is μ^*-measurable and $\mu = \widetilde{\mu}$ on $\mathcal{A}$ (see [Fol] prop. (1.13)). In particular $\mu([\alpha]) = 2^{-2^k}$ if $\alpha \in \Omega^k$. Later, I will construct such measures without going into such detail. This measure μ may be used to obtain a lower bound on $\mathcal{H}^{\psi^s}(\Omega)$.

Lemma 3.3.3 If (Ω, ρ) is infinite s-space, then $\mathcal{H}^{\psi^s}(\Omega) \geq 1/2$.

Proof: Let $U \subset \Omega$ satisfy $r^{k+1} < \text{diam}(U) \leq r^k$. Since $\text{diam}(U) > r^{k+1}$, there are $\sigma, \tau \in U$ so that $\sigma_{k+1} \neq \tau_{k+1}$. So, in fact, $\text{diam}(U) = r^k$. If $\sigma \in U$ and $\rho(\sigma, \tau) \leq r^k$, then $\tau_i = \sigma_i$ for $i = 1, \ldots, k$. So $U \subset [\sigma|_k]$. Thus

$$\mu(U) \leq \mu([\sigma|_k]) = 2^{-2^k} = 2^{-(1/r^k)^s} = \psi^s(\text{diam}(U))$$

and $\mathcal{H}^{\psi^s}(\Omega) \geq \mu(\Omega) = 1/2$, by lemma 3.3.2.□

Putting together lemmas 3.3.1 and 3.3.3 we get

Theorem 3.3.1 If (Ω, ρ) is infinite s-space, then $\mathcal{H}^{\psi^s}(\Omega) = 1/2$.

In fact, these arguments may be strengthened to show that $\mathcal{H}^{\psi^s} = \mu$ on Ω.

Lemma 3.3.2 is good but not quite powerful enough for some of my purposes. The next lemma is similar, but more widely applicable.

Lemma 3.3.4 For a separable metric space X with a positive Borel measure μ, $x \in X$, and $\delta > 0$ let

$$\mu_\delta(x) = \sup\{\mu(U) : x \in U \text{ and } \text{diam}(U) \leq \delta\}.$$

Let $\delta_k \searrow 0$. Suppose that $\varphi, \psi \in \Phi$ satisfy

$$\max_{k \in \mathbb{N}} \frac{\varphi(\delta_k)}{\psi(\delta_{k+1})} < A < \infty. \tag{3.2}$$

Let $E \subset X$ be a Borel set which satisfies

$$\overline{D}_\mu^\varphi(x, (\delta_k)) \equiv \limsup_{k \to 0} \frac{\mu_{\delta_k}(x)}{\varphi(\delta_k)} < M < \infty \text{ for every } x \in E.$$

Then $\mathcal{H}^\psi(E) = \infty$.

Proof: Let $k_0 \in \mathbb{N}$ and choose $0 < \varepsilon < \delta_{k_0}$. Let

$$E_{k_0} = \{x \in E : \mu_{\delta_k}(x) < M\varphi(\delta_k) \text{ for every } k \geq k_0\}.$$

Note that $\cup_{k_0=1}^{\infty} E_{k_0} = E$. Suppose that $\mathcal{C}$ is an ε-cover of E and so of E_{k_0}. For $k \geq k_0$ write

$$\mathcal{C}_k = \{U \in \mathcal{C} : \delta_{k+1} < \operatorname{diam}(U) \leq \delta_k\}.$$

I may assume that $\#(\mathcal{C}_k) < \infty$ for every k. Otherwise

$$\sum_{U \in C} \psi(\operatorname{diam}(U)) = \infty$$

and I'm done. For $U \in \mathcal{C}_k$ such that $U \cap E_{k_0} \neq \emptyset$ I have $\mu(U) < M\varphi(\delta_k)$. So

$$\begin{aligned}
\mu(E_{k_0}) &\leq \sum_{\substack{U \in \mathcal{C} \\ U \cap E_{k_0} \neq \emptyset}} \mu(U) = \sum_{k=k_0}^{\infty} \sum_{\substack{U \in \mathcal{C}_k \\ U \cap E_{k_0} \neq \emptyset}} \mu(U) \\
&\leq M \sum_{k=k_0}^{\infty} \#(\mathcal{C}_k)\varphi(\delta_k) \leq M \sum_{k=k_0}^{\infty} \sum_{U \in \mathcal{C}_k} \psi(\operatorname{diam}(U)) \frac{\varphi(\delta_k)}{\psi(\delta_{k+1})} \\
&= M \sum_{k=k_0}^{\infty} \left(\frac{\varphi(\delta_k)}{\psi(\delta_{k+1})} \sum_{U \in \mathcal{C}_k} \psi(\operatorname{diam}(U)) \right) \\
&\leq MA \left(\sum_{k=k_0}^{\infty} \sum_{U \in \mathcal{C}_k} \psi(\operatorname{diam}(U)) \right) \\
&= MA \left(\sum_{U \in \mathcal{C}} \psi(\operatorname{diam}(U)) \right).
\end{aligned}$$

Now $\mu(E_{k_0}) \to \mu(E)$ as $k_0 \to \infty$. Thus $\sum_{U \in \mathcal{C}} \psi(\operatorname{diam}(U)) > \mu(E_{k_0})/MA$ and $\mathcal{H}^{\psi}(E) > \mu(E)/MA$.$\square$

As an example, suppose that $\psi^s(t) = 2^{-(1/t)^s}$ and $\delta_k = cu^k$. Then for $s_1 < s_2$, I have

$$\frac{\psi^{s_2}(\delta_k)}{\psi^{s_1}(\delta_{k+1})} = 2^{-(\frac{1}{cu^k})^{s_2} + (\frac{1}{cu^{k+1}})^{s_1}}$$

$$= 2^{-(\frac{1}{cu^k})^{s_2}\left(1-(cu^k)^{s_2}(\frac{1}{cu^{k+1}})^{s_1}\right)}$$

$$= 2^{-(\frac{1}{cu^k})^{s_2}\left(1-(\frac{1}{u})^{s_1}(cu^k)^{s_2-s_1}\right)} \leq 2^{-\frac{1}{2}(\frac{1}{cu^k})^{s_2}}$$

for large k. This last term approaches zero as $k \to \infty$. So given a Borel set $E \subset X$, to show that $\mathcal{H}^{\psi^{s_1}}(E) = \infty$, it suffices to find a positive Borel measure μ on E, $M > 0$, and an $s_2 > s_1$ such that

$$\overline{D}_{\mu}^{\psi^{s_2}}(x, (cu^k)) < M$$

for every $x \in E$. In fact, this shows that $\dim(E) \succ \psi^{s^1}$, since if $s_1 < s' < s_2$, then $\mathcal{H}^{\psi^{s'}}(E) = \infty$ implying that $\mathcal{H}^{\psi^{s_1}}(E)$ is non-σ-finite.

3.4 s-Nested Packings

Part of the importance of sequence space is that it may be used to model many other spaces. In [Ed1], for example, self-similar sequence spaces are used in the study of self-similar sets in $\mathbb{R}^n$. In this section I will define a condition on a closed subset E of a complete separable metric space X which allows the construction of a subset $E' \subset E$ which is Lipeomorphic to a certain sequence space. This result will be used later to transfer results from sequence space to more general spaces.

Now let E be as above, fix $c, s > 0$, $\varepsilon \in (0, 1/4)$, and $m > (1/\varepsilon)^s + 1$. Let $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$ with the metric d given by $r(\alpha) = c\varepsilon^n$ for every $\alpha \in \Omega^n$. An *s-nested packing* of E will be a collection of closed balls $\{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha \in \Omega^*}$ satisfying:

1. $x_\alpha \in E$ for every $\alpha \in \Omega^*$

2. $B_{c\varepsilon^n}(x_\alpha) \cap B_{c\varepsilon^n}(x_\beta) = \emptyset$ for distinct $\alpha,\ \beta \in \Omega^n$.

3. $B_{c\varepsilon^{|\alpha|}}(x_\alpha) \subset B_{c\varepsilon^{|\alpha^-|}/4}(x_{\alpha^-})$ for every $\alpha \in \Omega^*$.

This definition also depends on c, ε, and m, however the important parameter is s because dimensional bounds given later will be in terms of s. If E has such an s-nested packing, then let $E' = \cap_{n=1}^{\infty} \cup_{\alpha \in \Omega^n} B_{c\varepsilon^{|\alpha|}}(x_\alpha)$. Define a map $g : \Omega \to E'$ by $g(\omega) = \cap_{n=1}^{\infty} B_{c\varepsilon^n}(x_{\omega|_n})$.

Lemma 3.4.1 The map g above is bi-Lipschitz.

Proof: Let $\omega_1,\ \omega_2 \in \Omega$ satisfy $d(\omega_1, \omega_2) = c\varepsilon^n$. Then $g(\omega_1),\ g(\omega_2) \in B_{c\varepsilon^n}(x_{\omega_1|_n})$, so $\rho(g(\omega_1), g(\omega_2)) \leq 2c\varepsilon^n$. For the lower bound, note that $g(\omega_i) \in B_{c\varepsilon^{n+1}/4}(x_{\omega_i|_{n+1}})$ for $i = 1, 2$ and $B_{c\varepsilon^{n+1}}(x_{\omega_1|_{n+1}}) \cap B_{c\varepsilon^{n+1}}(x_{\omega_2|_{n+1}}) = \emptyset$. So

$$
\begin{aligned}
c\varepsilon^{n+1} &\leq \rho(x_{\omega_1|_{n+1}}, x_{\omega_2|_{n+1}}) \\
&\leq \rho(x_{\omega_1|_{n+1}}, g(\omega_1)) + \rho(g(\omega_1), g(\omega_2)) + \rho(g(\omega_1), x_{\omega_2|_{n+2}}) \\
&\leq \frac{c\varepsilon^{n+1}}{4} + \rho(g(\omega_1), g(\omega_2)) + \frac{c\varepsilon^{n+1}}{4}.
\end{aligned}
$$

So $\rho(g(\omega_1), g(\omega_2)) \geq \frac{c\varepsilon^{n+1}}{2} = \frac{\varepsilon}{2} c\varepsilon^n$.$\square$

Next, I would like to show that this condition is non-vacuous by constructing an s-nested packing for a self similar set $E \subset X$. Self similar sets are obtained as follows: Let $m_1 \in \mathbb{N}$ and for $i = 1, \ldots, m_1$ let $f_i : X \to X$ be a similarity with ratio $r_i \in (0, 1)$. This means that for every $x,\ y \in X$ we have $\rho(f_i(x), f_i(y)) = r_i \rho(x, y)$. In this situation there exists a unique non-empty compact set $E \subset X$ such that $E = \cup_{i=1}^{m_1} f_i(E)$. A set obtained this way is said to be *self-similar*. [Ed1] has more details.

I will be using two sequence spaces Ω_1 and Ω_2 to analyze the set E. The first one is the self-similar sequence space $\Omega_1 = \{1,\dots,m_1\}^{\mathbb{N}}$ with metric d_1 given by the ratio list $(r_i)_{i=1}^{m_1}$ corresponding to the contraction ratios of $(f_i)_{i=1}^{m_1}$. Given $\alpha = (\alpha_1,\dots,\alpha_n) \in \Omega_1^n$, abbreviate $f_{\alpha_1} \circ \cdots \circ f_{\alpha_n}(E)$ by $\alpha(E)$. I will construct the s-nested packing in the metric space (E,ρ) rather than (X,ρ). The reason for this is because for $x \in E$, $\alpha \in \Omega_1^*$, and $\varepsilon > 0$ I have $\alpha(B_\varepsilon(x)) = B_{r(\alpha)\varepsilon}(\alpha(x))$ as long as only balls in (E,ρ) are considered. This is due to the invariance of E under the transformations $(f_i)_{i=1}^{m_1}$ and not generally true in the larger metric space (X,ρ).

Now let $s > 0$, let $r = \min\{r_i\}_{i=1}^{m_1}$, and let $c = \frac{8}{r}\max\{\mathrm{diam}(E),1\}$. Fix $\delta \in (0, \min\{\frac{1}{4}, \frac{1}{4}\mathrm{diam}(E)\})$ such that $N_{2\delta}(E) > (c/\delta)^s + 1$. Such a δ certainly exists if $\Delta(E) \succ t^s$. Let $\varepsilon = \delta/c$ and let $m_2 = N_{2\delta}(E)$. The other sequence space of interest is $\Omega_2 = \{1,\dots,m_2\}^{\mathbb{N}}$ with metric d_2 given by $r(\beta) = c\varepsilon^n$ for $\beta \in \Omega_2^n$.

I will construct an s-nested packing of E for the choices above. Choose $x_\Lambda \in E$ arbitrarily. This gives $B_c(x_\Lambda)$. The existence of $\{B_{c\varepsilon}(x_\beta)\}_{\beta \in \Omega_2^1}$ is guaranteed by the fact that $N_{2\delta}(E) > (c/\delta)^s + 1$ since $\delta = c\varepsilon$. The construction will proceed by induction on the length of β. Suppose that $B_{c\varepsilon^{|\beta|}}(x_\beta)$ have been defined for $|\beta| \le n$. For $\beta \in \Omega_2^n$, choose $\alpha_\beta \in \Omega_1^*$ such that $x_\beta \in \alpha_\beta(E)$, and

$$\mathrm{diam}(\alpha_\beta(E)) \le \frac{1}{8}\frac{\delta^n}{c^{n-1}} < \mathrm{diam}(\alpha_\beta^-(E)).$$

Since $r = \min\{r_i\}$ we have:

$$\mathrm{diam}(\alpha_\beta(E)) \ge \frac{r}{8}\frac{\delta^n}{c^{n-1}} = \frac{r}{8\mathrm{diam}(E)}\frac{\delta^n}{c^{n-1}}\mathrm{diam}(E) \ge \frac{\delta^n}{c^n}\mathrm{diam}(E).$$

So $N_{2\delta(\delta^n/c^n)}(\alpha_\beta(E)) \ge N_{2\delta}(E) = m_2$. Thus $\alpha_\beta(E)$ may be packed with m_2 balls of radius $\delta^{n+1}/c^n = c\varepsilon^{n+1}$ to continue the induction.

For a useful generalization, note that if $f : X \to X$ is a bi-Lipschitz map satisfying $\rho(f(x), f(y)) \geq r\rho(x, y)$ for every $x, y \in X$, then $f(B_\varepsilon(x)) \supset B_{r\varepsilon}(f(x))$. So if $f : E \to F$ is a bi-Lipschitz bijection and E has an s-nested packing $\{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha\in\Omega^*}$, then f induces an s-nested packing of F namely $\{B_{cr\varepsilon^{|\alpha|}}(f(x_\alpha))\}_{\alpha\in\Omega^*}$. Putting all this together we obtain:

Theorem 3.4.1 If $E \subset X$ has a subset which is Lipeomorphic to a self-similar set F satisfying $\Delta(F) \succ t^s$, then E has an s-nested packing.

Finally, I will have occasion to extract a packing $\mathcal{B} \subset \{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha\in\Omega^*}$ from a given s-nested packing satisfying the condition described in the following lemma.

Lemma 3.4.2 Given an s-nested packing there is a packing $\mathcal{B} \subset \{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha\in\Omega^*}$ such that for every $n \in \mathbb{N}$,

$$\#(\{\alpha \in \Omega^n : B_{c\varepsilon^n}(x_\alpha) \in \mathcal{B}\}) = (m-1)^{n-1}.$$

Proof: Choose one $\alpha \in \Omega^1$ arbitrarily and put $B_{c\varepsilon}(x_\alpha) \in \mathcal{B}$. Continuing recursively, suppose that $B_{c\varepsilon^{|\alpha|}}(x_\alpha)$ have been chosen for $|\alpha| \leq n$ such that

$$\#(\{\alpha \in \Omega^k : B_{c\varepsilon^k}(x_\alpha) \in \mathcal{B}\}) = (m-1)^{k-1}$$

for each $k = 1, \ldots, n$. A given $B_{c\varepsilon^k}(x_\alpha) \in \mathcal{B}$ where $\alpha \in \Omega^k$ contains m^{n+1-k} balls $B_{c\varepsilon^{n+1}}(x_\beta)$ where $\beta \in \Omega^{n+1}$. For fixed k, $\mathcal{B}$ contains $(m-1)^{k-1}$ such balls $B_{c\varepsilon^k}(x_\alpha)$. Thus for this fixed k,

$$\#(\{\beta \in \Omega^{n+1} : B_{c\varepsilon^{n+1}}(x_\beta) \subset B_{c\varepsilon^k}(x_\alpha) \in \mathcal{B}\}) = m^{n+1-k}(m-1)^{k-1}.$$

So then the number of $\beta \in \Omega^{n+1}$ such that $B_{c\varepsilon^{n+1}}(x_\beta)$ is contained in no $B_{c\varepsilon^k}(x_\alpha) \in \mathcal{B}$ for any $k = 1, \ldots, n$ is

$$\begin{aligned} m^{n+1} - \sum_{k=1}^{n}(m-1)^{k-1}m^{n+1-k} &= m^{n+1}\left(1 - \frac{1}{m}\sum_{k=1}^{n}\left(\frac{m-1}{m}\right)^{k-1}\right) \\ &= m^{n+1}\left(1 - \frac{1}{m}\left(\frac{(\frac{m-1}{m})^n - 1}{\frac{m-1}{m} - 1}\right)\right) \\ &= m^{n+1}\left(1 + \frac{(m-1)^n}{m^n} - 1\right) \\ &= m(m-1)^n > (m-1)^n. \end{aligned}$$

Thus the induction may continue.□

3.5 A Cartesian Product Example

The Hausdorff measure and dimension were defined in 1918. The related theory is now greatly developed and the definition has withstood the test of time. In contrast, the packing measure and dimension have developed relatively recently. If one looks at the recent references [Ed1], [TayTr], or [Tr], one finds a slightly different definition of packing measure based on the diameter of a ball rather than the radius of a ball, as I have done. Specifically, for $\varphi \in \Phi$ construct a *diameter based packing measure* Q^φ on a separable metric space (X, ρ) as follows: For $E \subset X$ an ε-*packing* of E (for this section only) is a finite or countable collection of closed balls $\{B_{r_i}(x_i)\}_i$ with centers in E such that $\operatorname{diam}(B_{r_i}(x_i)) \leq \varepsilon$ for every i. Let

$$\widetilde{Q}^\varphi_\varepsilon(E) = \sup\{\sum_i \varphi(\operatorname{diam}(B_{r_i}(x_i))) : \{B_{r_i}(x_i)\}_i \text{ is an } \varepsilon\text{-packing of } E\},$$

$$\widetilde{Q}^{\varphi}(E) = \lim_{\varepsilon \to 0} \widetilde{Q}^{\varphi}_{\varepsilon}(E),$$

and

$$Q^{\varphi}(E) = \inf\{\sum_{i=1}^{\infty} \widetilde{Q}^{\varphi}(E_i) : E \subset \cup_{i=1}^{\infty} E_i\}.$$

These definitions are very similar to the definitions for $\mathcal{P}^{\varphi}$ given in section 2.1.2, except that twice the radius of a ball $2r_i$ is replaced by the diameter of a ball $\mathrm{diam}(B_{r_i}(x_i))$. Q^{φ} is a reasonable measure which is clearly equal to $\mathcal{P}^{\varphi}$ on Euclidean space $\mathbb{R}^n$. In [Cut], however, it is argued that the radius based definition more closely preserves the desirable properties of packing measure and dimension on Euclidean space. For example, she mentions that the value of the pre-measure $\widetilde{Q}^{\varphi}_{\varepsilon}$ is sensitive to whether closed balls or open balls are used, while $\widetilde{\mathcal{P}}^{\varphi}_{\varepsilon}$ will not be affected by this choice. The freedom to pack with either closed balls or open balls is frequently convenient in proofs. In this section, I intend to give an example involving cartesian products supporting the view that the radius based definition is preferable for general metric spaces.

One very nice feature of the packing dimension in Euclidean space is its behavior with respect to Cartesian products. Given metric spaces (X, ρ_x) and (Y, ρ_y), a metric ρ may be defined on the set $X \times Y = \{(x, y) : x \in X, y \in Y\}$ by

$$\rho((x_1, y_1), (x_2, y_2)) = \max\{\rho_x(x_1, x_2), \rho_y(y_1, y_2)\}.$$

Let $\varphi_s(t) = t^s$. Part of [Tr] theorem 5 states that if $\mathrm{Dim}(E) \preceq \varphi_{s_1}$ and $\mathrm{Dim}(F) \preceq \varphi_{s_2}$, then $\mathrm{Dim}(E \times F) \preceq \varphi_{s_1+s_2}$ when E and F are subsets of Euclidean space. There is a similar statement made in [Weg] on page 68 for the upper entropy dimension valid for arbitrary metric spaces. By the comparison theorem 2.2.2, the radius based packing dimension behaves in a similar manner. The following example shows that this is not the case for the diameter based packing measure:

Example 3.5.1 Let Ω_1 be the sequence space as described in section 3.1 corresponding to the sequence of natural numbers $(a_n)_{n=1}^{\infty}$ with diameter function $r : \Omega_1^* \to (0, 1]$ given by $r(\alpha) = \prod_{i=1}^{k} \frac{1}{a_i}$ for $\alpha \in \Omega_1^k$ and $r(\Lambda) = 1$. Define Ω_2 similarly, but with the sequence $(b_n)_{n=1}^{\infty}$. Let $\varphi_1(t) = t$. Then $Q^{\varphi_1}(\Omega_1) \leq 1$ and $Q^{\varphi_1}(\Omega_2) \leq 1$. But given any $\varphi \in \Phi$ the sequences $(a_n)_n$ and $(b_n)_n$ may be chosen to make $Q^{\varphi}(\Omega_1 \times \Omega_2) = \infty$.

Proof: To show that $Q^{\varphi_1}(\Omega_1) \leq 1$, I will use the notion of a refinement of a packing. Given a packing $\mathcal{B}_1$ of a sequence space Ω, a *refinement* $\mathcal{B}_2$ of $\mathcal{B}_1$ is a packing such that for every $[\beta] \in \mathcal{B}_2$, there is an $[\alpha] \in \mathcal{B}_1$ such that $[\beta] \subset [\alpha]$. Note that any packing of Ω is a refinement of the trivial packing $[\Lambda]$.

Next note that if $\alpha \in \Omega_1^k$ and A denotes the set of all children of α, then

$$\sum_{\beta \in A} \operatorname{diam}([\beta]) = a_{k+1} \prod_{i=1}^{k+1} \frac{1}{a_i} = \prod_{i=1}^{k} \frac{1}{a_i} = \operatorname{diam}([\alpha]).$$

It follows that if $\mathcal{B}_1$ is a packing of Ω_1 and $\mathcal{B}_2$ is a refinement of $\mathcal{B}_1$, then

$$\sum_{[\beta] \in \mathcal{B}_2} \operatorname{diam}([\beta]) \leq \sum_{[\alpha] \in \mathcal{B}_1} \operatorname{diam}([\alpha]).$$

In particular, any packing $\mathcal{B}$ of Ω_1 satisfies $\sum_{[\alpha] \in \mathcal{B}} \operatorname{diam}([\alpha]) \leq \operatorname{diam}([\Lambda]) = 1$. So $Q^{\varphi}(\Omega_1) \leq \widetilde{Q}^{\varphi}(\Omega_1) \leq 1$. Similar considerations apply to Ω_2.

Now let $\varphi \in \Phi$. I will recursively define integers $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ so that $Q^{\varphi}(\Omega_1 \times \Omega_2) = \infty$. Let $a_1 = 1$, and choose $b_1 > \max\{a_1, 1/\varphi(a_1)\}$. Choose $a_2 \in \mathbb{N}$

such that $\frac{1}{a_1 a_2} < \frac{1}{b_1} < \frac{1}{a_1}$, then choose $b_2 \geq 2/(a_1 a_2 b_1 \varphi(\frac{1}{a_1 a_2}))$ such that $\frac{1}{b_1 b_2} < \frac{1}{a_1 a_2}$. Suppose that $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ have been chosen so that for every $k = 2, \ldots, n$ we have

$$\prod_{i=1}^{k} \frac{1}{a_i} < \prod_{i=1}^{k-1} \frac{1}{b_i} < \prod_{i=1}^{k-1} \frac{1}{a_i} \tag{3.3}$$

and $(\prod_{i=1}^{k} a_i b_i)\varphi(\prod_{i=1}^{k} \frac{1}{a_i}) \geq k$. Choose a_{n+1} such that $\prod_{i=1}^{n+1} \frac{1}{a_i} < \prod_{i=1}^{n} \frac{1}{b_i}$, then choose

$$b_{n+1} \geq \frac{n+1}{(\prod_{i=1}^{n+1} a_i)(\prod_{i=1}^{n} b_i)\varphi(\prod_{i=1}^{n+1} \frac{1}{a_i})}$$

such that $\prod_{i=1}^{n+1} \frac{1}{b_i} < \prod_{i=1}^{n+1} \frac{1}{a_i}$.

Now if $\Omega_1 \times \Omega_2 = \cup_{n=1}^{\infty} K_n$ where each K_n is closed, then by Baire category there are $k, n \in \mathbb{N}$ and $\alpha \in \Omega_1^k$, $\beta \in \Omega_2^k$ such that $[\alpha] \times [\beta] \subset K_n$. Thus to show that $Q^{\varphi}(\Omega_1 \times \Omega_2) = \infty$, it suffices to show that $\widetilde{Q}^{\varphi}([\alpha] \times [\beta]) = \infty$ for $\alpha \in \Omega_1^k$, $\beta \in \Omega_2^k$.

So suppose that $\alpha \in \Omega_1^k$, $\beta \in \Omega_2^k$. Let $n > k$ and let α' and β' be descendants of α and β respectively of length n. Then $\mathrm{diam}([\alpha']) = \prod_{i=1}^{n} \frac{1}{a_i}$ and $\mathrm{diam}([\beta']) = \prod_{i=1}^{n} \frac{1}{b_i}$. From the condition in equation 3.3, it follows that if $(\omega_1, \sigma_1), (\omega_2, \sigma_2) \in [\alpha'] \times [\beta']$, then $\rho((\omega_1, \sigma_1), (\omega_2, \sigma_2)) \leq \prod_{i=1}^{n} \frac{1}{a_i}$. Conversely, any point $(\omega_3, \sigma_3) \in \Omega_1 \times \Omega_2 \setminus [\alpha'] \times [\beta']$ satisfies $\rho((\omega_1, \sigma_1), (\omega_3, \sigma_3)) > \prod_{i=1}^{n} \frac{1}{a_i}$. So $[\alpha'] \times [\beta']$ is a ball (closed and open) of diameter $\prod_{i=1}^{n} \frac{1}{a_i}$. The collection of all such $[\alpha'] \times [\beta']$ forms a $\prod_{i=1}^{n} \frac{1}{a_i}$-packing of $[\alpha] \times [\beta]$ with $\prod_{i=k+1}^{n} a_i b_i$ elements. So for $\varepsilon = \prod_{i=1}^{n} \frac{1}{a_i}$,

$$\widetilde{Q}_{\varepsilon}^{\varphi}([\alpha] \times [\beta]) \geq (\prod_{i=k+1}^{n} a_i b_i)\varphi(\prod_{i=1}^{n} \frac{1}{a_i}) \geq \frac{n}{\prod_{i=1}^{k} a_i b_i} \to \infty$$

as $n \to \infty$. Thus $\widetilde{Q}^{\varphi}([\alpha] \times [\beta]) = \infty$ and $Q^{\varphi}(\Omega_1 \times \Omega_2) = \infty$.$\square$

CHAPTER IV

Dimensions of Hyperspaces

Given a metric space, (X, ρ), a hyperspace associated with X is a metric space whose elements are subsets of X. In this chapter I will discuss the relationship between the dimension of a metric space and various hyperspaces.

4.1 Entropy Dimensions of the Space $\mathcal{K}(X)$

Given a metric space (X, ρ), let $\mathcal{K}(X)$ denote the set of non-empty, compact subsets of X. Endow $\mathcal{K}(X)$ with a metric $\widetilde{\rho}$ as follows: For $A, B \subset X$ let

$$\operatorname{dist}(A, B) = \inf\{\rho(x, y) : x \in A,\ y \in B\}.$$

Then for $A, B \in \mathcal{K}(X)$ let

$$\widetilde{\rho}(A, B) = \max\{\sup_{x \in A}\{\operatorname{dist}(x, B)\}, \sup_{y \in B}\{\operatorname{dist}(y, A)\}\}.$$

The metric $\widetilde{\rho}$ so defined is generally called the *Hausdorff metric.* Note that a tilde will frequently indicate that I am working in the hyperspace. The space $(\mathcal{K}(X), \widetilde{\rho})$ inherits many nice features from (X, ρ). For example, $\mathcal{K}(X)$ is complete whenever X

is complete and $\mathcal{K}(X)$ is compact whenever X is compact. For a proof of these facts, including that $\widetilde{\rho}$ is a metric, see [Ed1] section 2. 4.

The following lemmas describe the relationship between $\Delta(E)$ and $\Delta(\mathcal{K}(E))$. Note that all that follows holds for δ and $\widehat{\delta}$ with very similar proofs.

Lemma 4.1.1 For totally bounded $E \subset X$, $\Delta(E) \succ \varphi$ implies $\Delta(\mathcal{K}(E)) \succ 2^{-1/\varphi}$.

Proof: For $\varepsilon > 0$, let $\mathcal{B}_\varepsilon = \{B_{\varepsilon/2}(x_i)\}_{i=1}^{N_\varepsilon(E)}$ be an ε-packing of E with $N_\varepsilon(E)$ balls of radius $\varepsilon/2$. If $F \subset \{x_i\}_{i=1}^{N_\varepsilon(E)}$ is non-empty, then $F \in \mathcal{K}(E)$ and we may consider $\widetilde{B}_{\varepsilon/2}(F) \subset \mathcal{K}(E)$ the ball of radius $\varepsilon/2$ centered on F. Note that

$$\widetilde{B}_{\varepsilon/2}(F) = \{C \in \mathcal{K}(E) : C \subset \cup_{x_i \in F} B_{\varepsilon/2}(x_i) \text{ and } C \cap B_{\varepsilon/2}(x_i) \neq \emptyset \, \forall x_i \in F\}.$$

From this it is easy to see that given distinct, non-empty F_1, $F_2 \subset \{x_i\}_{i=1}^{N_{\varepsilon/2}(E)}$ we have $\widetilde{B}_{\varepsilon/2}(F_1) \cap \widetilde{B}_{\varepsilon/2}(F_2) = \emptyset$. So $N_\varepsilon(\mathcal{K}(E)) \geq 2^{N_\varepsilon(E)} - 1$ and

$$\begin{aligned} N_\varepsilon(\mathcal{K}(E))2^{-1/\varphi(\varepsilon)} &\geq (2^{N_\varepsilon(E)} - 1)2^{-1/\varphi(\varepsilon)} = 2^{N_\varepsilon(E) - \frac{1}{\varphi(\varepsilon)}} - 2^{\frac{-1}{\varphi(\varepsilon)}} \\ &= 2^{\frac{N_\varepsilon(E)\varphi(\varepsilon)-1}{\varphi(\varepsilon)}} - 2^{\frac{-1}{\varphi(\varepsilon)}}. \end{aligned}$$

Thus

$$\limsup_{\varepsilon\to 0} N_\varepsilon(\mathcal{K}(E))2^{-1/\varphi(\varepsilon)} \geq \limsup_{\varepsilon\to 0} \left(2^{\frac{N_\varepsilon(E)\varphi(\varepsilon)-1}{\varphi(\varepsilon)}} - 2^{\frac{-1}{\varphi(\varepsilon)}}\right) = \infty,$$

since $\limsup_{\varepsilon\to 0} N_\varepsilon(E)\varphi(\varepsilon) = \infty$. So $\Delta(\mathcal{K}(E)) \succ 2^{-1/\varphi}$.$\Box$

For the reverse inequality it will be useful to define the quantity M_ε by

$$M_\varepsilon(E) = \text{ min number of sets of diameter } \leq \varepsilon \text{ needed to cover } E.$$

M_ε is related to N_ε as per the following lemma.

Lemma 4.1.2 For totally bounded E and $\varepsilon > 0$, $N_\varepsilon(E) \leq M_{\varepsilon/2}(E) \leq N_{\varepsilon/4}(E)$.

Proof: The first inequality is because any two closed disjoint balls of radius $\varepsilon/2$ must have centers separated by more than $\varepsilon/2$. Thus an $\varepsilon/2$-cover requires at least one set for each element of an ε-packing.

The second inequality follows from the fact that a maximal packing by $\varepsilon/8$-balls induces an $\varepsilon/2$-cover by doubling the radius of each of the balls.$\square$

Lemma 4.1.3 For E totally bounded, $\Delta(E) \prec \varphi(t)$ implies $\Delta(\mathcal{K}(E)) \prec 2^{-1/\varphi(t/4)}$.

Proof: Let $\mathcal{B}_\varepsilon = \{B_{\varepsilon/2}(x_i)\}_{i=1}^{N_\varepsilon(E)}$ be an ε-packing of E. Then $\widetilde{\mathcal{B}}_\varepsilon = \{\widetilde{B}_\varepsilon(F)\}_{F\subset\{x_i\}}$ forms a 2ε-cover of $\mathcal{K}(E)$ with $2^{N_\varepsilon(E)} - 1$ elements. So

$$N_{4\varepsilon}(\mathcal{K}(E)) \leq M_{2\varepsilon}(\mathcal{K}(E)) \leq 2^{N_\varepsilon(E)} - 1$$

for every $\varepsilon > 0$. So

$$\begin{aligned} N_{4\varepsilon}(\mathcal{K}(E))2^{-1/\varphi(\varepsilon)} &\leq (2^{N_\varepsilon(E)} - 1)2^{-1/\varphi(\varepsilon)} \\ &= 2^{\frac{N_\varepsilon(E)\varphi(\varepsilon)-1}{\varphi(\varepsilon)}} - 2^{-1/\varphi(\varepsilon)} \to 0 \end{aligned}$$

as $\varepsilon \to 0$. Thus $\Delta(\mathcal{K}(E)) \prec 2^{-1/\varphi(t/4)}$.$\square$

It is natural to ask if $\Delta(E) \asymp \varphi$ implies $\Delta(\mathcal{K}(E)) \asymp 2^{-1/\varphi}$. The answer is no, basically because the $\asymp$ relation is too sensitive. For example, if $I \subset \mathbb{R}$ is a closed interval of length ℓ, then the optimal ε-packing of I consists of $[\ell/\varepsilon]$ balls with centers separated by ε. So $N_\varepsilon(I) = [\ell/\varepsilon]$ and $\lim_{\varepsilon\to 0} \varepsilon N_\varepsilon(I) = \ell$. So $\Delta(I) \asymp \varepsilon$ for any bounded interval I. But, as we see in the proofs of lemmas 4.1.1 and 4.1.3,

$$2^{N_\varepsilon(E)} - 1 \leq N_\varepsilon(\mathcal{K}(E)) \leq 2^{N_{\varepsilon/4}(E)} - 1.$$

So

$$N_\varepsilon(\mathcal{K}(I))2^{-1/\varepsilon}\begin{cases} \geq (2^{[\ell/\varepsilon]}-1)2^{-1/\varepsilon} \to \infty \text{ if } \ell > 1 \\ \leq (2^{[4\ell/\varepsilon]}-1)2^{-1/\varepsilon} \to 0 \ \text{ if } \ell < \frac{1}{4}. \end{cases}$$

And

$$\Delta(\mathcal{K}(I))\begin{cases} \succ 2^{-1/\varepsilon} \text{ if } \ell > 1 \\ \prec 2^{-1/\varepsilon} \text{ if } \ell < \frac{1}{4}. \end{cases}$$

The next step is to pass to the entropy dimensions.

Theorem 4.1.1 If $E \subset X$ is σ-compact and $\widehat{\Delta}(E) \succ \varphi$, then $\widehat{\Delta}(\mathcal{K}(E)) \succ 2^{-1/\varphi}$.

Proof: Suppose that $\widehat{\Delta}(E) \succ \varphi$ and E is compact. The extension of the theorem to σ-compact sets is straightforward as any such set contains a compact subset E satisfying $\widehat{\Delta}(E) \succ \varphi$ by σ-stability. I will need to highlight a subset of E with a certain regularity property. Let

$$E_\varphi = \{x \in E : \widehat{\Delta}(E \cap B_r(x)) \succ \varphi \, \forall \, r > 0\}.$$

$E \backslash E_\varphi$ is open in E practically by definition. So E_φ is closed in E and, therefore, σ-compact. Let $x \in E_\varphi$ and let $r > 0$. I claim that $\widehat{\Delta}(E_\varphi \cap B_r(x)) \succ \varphi$. (This is the needed regularity property of E_φ.) $\widehat{\Delta}(E \cap B_r(x)) \succ \varphi$ by definition. For every $y \in (E \backslash E_\varphi) \cap B_r(x)$, choose an r_y such that $\widehat{\Delta}(E \cap B_{r_y}(y)) \not\succ \varphi$. Then $\{B_{r_y}(y)\}_{y \in (E \backslash E_\varphi) \cap B_r(x)}$ is an open cover of the σ-compact set $(E \backslash E_\varphi) \cap B_r(x)$. So there is a countable subcover $\{B_{r_{y_k}}(y_k)\}_{k=1}^{\infty}$. Since

$$E \cap B_r(x) = (\cup_{k=1}^{\infty} B_{r_{y_k}}(y_k) \cap E) \cup (E_\varphi \cap B_r(x))$$

and $\widehat{\Delta}(B_{r_{y_k}}(y_k) \cap E) \not\succ \varphi \, \forall \, k$, it must be that $\widehat{\Delta}(E_\varphi \cap B_r(x)) \succ \varphi$ by σ-stability of $\widehat{\Delta}$.

I'll now show that $\widehat{\Delta}(\mathcal{K}(E_\varphi)) \succ 2^{-1/\varphi}$ from which it easily follows that $\widehat{\Delta}(\mathcal{K}(E)) \succ 2^{-1/\varphi}$. Suppose that $\mathcal{K}(E_\varphi) = \cup_1^\infty \widetilde{K}_n$. It may be assumed that each $\widetilde{K}_n$ is closed. By the Baire category theorem, one of the $\widetilde{K}_n$'s is somewhere dense. This means that there is a set $D \in \widetilde{K}_n$ and an $r > 0$ such that $\widetilde{B}_r(D) \subset \widetilde{K}_n$. Let $x \in D$ and define

$$\widetilde{A}_{x,r} = \{C \in \widetilde{K}_n : C = (D \backslash B_r) \cup F, \text{ where } F \subset B_{r/2}(x) \cap E_\varphi \text{ is compact}\}.$$

$\widetilde{A}_{x,r} \subset \widetilde{K}$ is naturally isometric to $\mathcal{K}(E_\varphi \cap B_{r/2}(x))$ which satisfies $\Delta(\mathcal{K}(E_\varphi \cap B_{r/2}(x))) \succ 2^{-1/\varphi}$, since $\Delta(E_\varphi \cap B_{r/2}(x)) \succ \varphi$. Therefore, $\Delta(\widetilde{K}_n) \succ 2^{-1/\varphi}$ and $\widehat{\Delta}(\mathcal{K}(E_\varphi)) \succ 2^{-1/\varphi}$.$\square$

The next example shows that the converse inequality does not hold.

Example 4.1.1 Recall the situation from example 2.1.1: Let $a_n \in \varphi^{-1}(\{1/n\})$ where $\varphi \in \Phi$. Let $X = \{x_0, x_1, x_2, \ldots, x_\infty\}$ be a countable set. Define a metric ρ on X by

$$\rho(x_n, x_m) = \begin{cases} a_n & \text{if } n \neq m = \infty \\ a_n + a_m & \text{if } \infty \neq n \neq m \neq \infty \\ 0 & \text{if } n = m. \end{cases}$$

Then $\widehat{\Delta}(X) \prec \psi$ for every $\psi \in \Phi$ as X is the countable union of singletons. But $\widehat{\Delta}(\mathcal{K}(X)) \succ 2^{-1/\varphi}$.

Proof: Write $\mathcal{K}(X) = \widetilde{K} \cup \widetilde{I}$ where

$$\widetilde{K} = \{C \in \mathcal{K}(X) : x_\infty \in C\} \quad \text{and} \quad \widetilde{I} = \{C \in \mathcal{K}(X) : x_\infty \notin C\}.$$

$\widetilde{I}$ consists of the countably many isolated points of $\mathcal{K}(X)$. So $\widetilde{K}$ is closed in $\mathcal{K}(X)$. Any decomposition of $\mathcal{K}(X)$ induces one of $\widetilde{K}$, so suppose that $\widetilde{K} = \cup_{n=1}^\infty \widetilde{K}_n$, where the $\widetilde{K}_n$ may be assumed closed. By the Baire Category Theorem, one of the $\widetilde{K}_n$ is somewhere dense. So there exists $n \in \mathbb{N}$, $C \in \widetilde{K}_n$, and $r > 0$ such that $\widetilde{B}_r(C) \subset \widetilde{K}_n$. Let

$$X_r = \{x \in X : \rho(x, x_\infty) \leq r\}, \quad C_r = \{x \in C : \rho(x, x_\infty) > r\},$$

and let

$$\widetilde{A} = \{D \in \mathcal{K}(E) : D = C_r \cup F \text{ where } F \in \mathcal{K}(X_r)\}.$$

Then $\widetilde{A} \subset \widetilde{B}_r(C) \subset \widetilde{K}_n$ and $\widetilde{A}$ is naturally isometric to $\mathcal{K}(X_r)$. Now $\Delta(X_r) \succ \varphi$ by example 2.2.1, so $\Delta(\mathcal{K}(X_r)) \succ 2^{-1/\varphi}$ by lemma 4.1.1. Thus $\widehat{\Delta}(\mathcal{K}(X)) \succ 2^{-1/\varphi}$.□

In spite of example 4.1.1 we do have the following:

Lemma 4.1.4 Suppose that E is totally bounded and $\Delta(E) \prec \varphi(t)$. Then $\widehat{\Delta}(E) \prec 2^{-1/\varphi(t/4)}$.

Proof: This immediate by lemma 4.1.3.□

Thus sets which are "regular enough" satisfy the expected type of upper bound.

Another interesting example is $\mathcal{K}([0,1])$. For $c > 0$, let $\varphi_c(t) = 2^{-\frac{c}{t}}$. In [Goo1], it is shown that $\dim(\mathcal{K}([0,1])) \prec \varphi_c$ for every $c > 0$. A careful reading of the proof there shows, in fact, $\widehat{\Delta}(\mathcal{K}([0,1])) \prec \varphi_c$ for every $c > 0$. I will investigate a converse to this statement using theorem 4.1.1 and the following lemma. Logarithms are to the base 2.

Lemma 4.1.5 Let $\varphi_c(t) = 2^{-\frac{c}{t}}$ and suppose that $\varphi \prec \varphi_c$ for every $c > 0$. Let $\psi(t) = -1/\log(\varphi(t))$. Then $\psi \prec t$.

Proof: Let $c > 0$ and choose $t_c > 0$ such that $0 < t < t_c$ implies $\frac{\varphi_c(t)}{\varphi(t)} \leq 1$. Then

$$\frac{t}{\psi(t)} = -t\log\varphi(t) \leq -t\log\varphi_c(t) = \log(2^{\frac{-c}{t}})^{-t} = c.$$

So $\frac{t}{\psi(t)} \to 0$ as $t \to 0$, since $c > 0$ is arbitrary.$\Box$

Corollary 4.1.1 $\widehat{\Delta}(\mathcal{K}([0,1])) \succ \varphi$, where φ is as above.

Proof: Since $\psi(t) \prec t$, I have $\widehat{\Delta}([0,1]) \succ \psi$ and so $\widehat{\Delta}(\mathcal{K}([0,1])) \succ 2^{-1/\psi} = \varphi$, by theorem 4.1.1.$\Box$.

4.2 Hausdorff Dimension of $\mathcal{K}(X)$

In this section I investigate conditions on E to ensure that $\dim(\mathcal{K}(E)) \sim 2^{-(1/\varepsilon)^s}$. Early results in this direction are in [Boa, Goo1, Goo2] which deal with $\mathcal{K}([0,1])$.

Calculations of Hausdorff dimensions are generally much more difficult than those for entropy dimensions. Assumptions made on E will be much more stringent. I begin with self-similar sequence space Ω and will then transfer the results to other sets modeled by Ω. In this section, $\Omega = (1,\ldots,m)^{\mathbb{N}}$ is a fixed self-similar sequence space with contraction ratio list $(r_1,\ldots,r_m)$ so that $\sum_1^m r_i^{s_0} = 1$.

Theorem 4.2.1 For $M > 0$ let $\varphi_M(\varepsilon) = 2^{-M(1/\varepsilon)^{s_0}}$. Then there exists an M large enough so that $\mathcal{H}^{\varphi_M}(\mathcal{K}(\Omega)) < \infty$.

Proof: Choose $0 < u < \min\{r_i\}$ so that $1/u^{s_0} = n \in \mathbb{N}$. For every $k \in \mathbb{N}$, let

$$L_k = \{\alpha \in \Omega^* : r(\alpha) \leq u^k < r(\alpha^-)\}.$$

Each $\alpha \in L_k$ satisfies

$$u^{k+1} < r(\alpha) \leq u^k$$

and

$$n^{-(k+1)} < \mathcal{H}^{s_0}([\alpha]) \le n^{-k} \tag{4.1}$$

by equation 3.1. Suppose that $\#(L_1) = L$. Then since each $\beta \in L_1$ satisfies $n^{-2} < \mathcal{H}^{s_0}([\beta]) \le n^{-1}$, for each $k \in \mathbb{N}$ the number of descendants of β in L_k cannot exceed n^k. So

$$Ln^{k-2} \le \#(L_k) \le Ln^k.$$

Let $A \subset L_k$ be nonempty. Associate with A a set $\widetilde{A} \subset \mathcal{K}(\Omega)$ defined by:

$$\widetilde{A} = \{C \in \mathcal{K}(\Omega) : \{\alpha \in L_k : [\alpha] \cap C \ne \emptyset\} = A\}.$$

Such a set $\widetilde{A}$ is called a *k-set* and satisfies $\mathrm{diam}(\widetilde{A}) \le u^k$. Since $\#(L_k) \le Ln^k$, there are no more than $2^{Ln^k} - 1$ such k-sets. This leads to the following estimate:

$$\begin{aligned} \mathcal{H}^{\varphi_M}_{u^k}(\mathcal{K}(\Omega)) &\le \left(2^{Ln^k} - 1\right) \varphi_M(u^k) \\ &\le 2^{Ln^k} 2^{-M(1/u^k)^s} = 2^{Ln^k} 2^{-Mn^k} \le 1 \end{aligned}$$

as long as $M \ge L$. Thus for $M \ge L$, I have $\mathcal{H}^{\varphi_M}(\mathcal{K}(\Omega)) \le 1$.□

Now for the lower bound, let $\psi^s(\varepsilon) = 2^{-(1/\varepsilon)^s}$ for $s > 0$.

Theorem 4.2.2 $\mathcal{H}^{\psi^s}(\mathcal{K}(\Omega)) > 0$, whenever $s < s_0$.

Proof: L_k, L, u, n, and k-sets are as in the preceding proof. Given $A \subset L_k$, define $\pi(A) = \#(A)/Ln^{k-2}$. Let $n' = 1/u^s < n$, choose $p \in (0, 1)$ so that $n' < pn$, then choose $j \in \mathbb{N}$ large enough so that $\left(\frac{n'}{pn}\right)^j < \frac{1}{n}$. I will construct a measure μ concentrated on those kj-sets $\widetilde{A}$ with $\pi(A) \ge p^{kj}/n^k$ so that μ satisfies $\overline{D}^{\psi^s}_\mu(x, (u^{kj+1})) \le 1$

for every $x \in \mathcal{K}(\Omega)$ implying the result by lemma 3.3.4. Recall that the definition of $\overline{D}_{\mu}^{\psi^s}(x, (u^{kj+1}))$ is given in lemma 3.3.4.

The measure μ will be constructed recursively. The empty word Λ is the only string of length 0 leading to the one 0-set $\widetilde{\Lambda} = \mathcal{K}(\Omega)$. Define $\mu(\mathcal{K}(\Omega)) = 1$. Fix $k \in \mathbb{N}$ and suppose that μ has been defined for all kj-sets $\widetilde{A}$ such that $\mu(\widetilde{A}) > 0$ only if $\pi(A) \geq p^{kj}/n^k$. If $\widetilde{A}$ is such a kj-set, then distribute $\mu(\widetilde{A})$ among all those $(k+1)j$-sets $\widetilde{B} \subset \widetilde{A}$ such that $\pi(B) \geq \frac{p^{(k+1)j}}{n^{k+1}}$. Such a set $\widetilde{B}$ will be called an eligible descendent of $\widetilde{A}$. I need a lower bound on the number of eligible descendants of $\widetilde{A}$. I have $\#(A) \geq \frac{p^{kj}}{n^k} L n^{kj-2}$, since $\pi(A) \geq \frac{p^{kj}}{n^k}$. If $\alpha \in A \subset L_{kj}$, then

$$n^{-(kj+1)} < \mathcal{H}^{s_0}([\alpha]) \leq n^{-kj},$$

by equation 4.1 while if $\beta \in L_{(k+1)j}$, then similarly

$$n^{-(k+1)j-1} < \mathcal{H}^{s_0}([\beta]) \leq n^{-(k+1)j}.$$

Thus if $L_{(k+1)j,\alpha}$ is the set of descendants of α in $L_{(k+1)j}$, then

$$n^{j-1} < \#(L_{(k+1)j,\alpha}) < n^{j+1}.$$

To form an eligible descendent $\widetilde{B} \subset \widetilde{A}$ proceed as follows: Take $\left[p^j \frac{p^{kj}}{n^k} L n^{kj-2}\right]$ of the α's $\in \widetilde{A}$ and choose all possible descendants β to form part of the set B. This guarantees that

$$\#(B) \geq \left[\frac{p^{(k+1)j}}{n^k} L n^{kj-2} n^{j-1}\right] = \left[\frac{p^{(k+1)j}}{n^k} L n^{(k+1)j-3}\right],$$

so that $\pi(B) \geq \frac{p^{(k+1)j}}{n^{k+1}}$ as required. I am now free to choose descendants of the remaining

$$\left[\frac{p^{kj}}{n^k}Ln^{kj-2}\right]-\left[\frac{p^{(k+1)j}}{n^k}Ln^{kj-2}\right]\geq(1-p^j)(\frac{p^{kj}}{n^k}Ln^{kj-2})-1$$

α's $\in A$ as I like. Since each $\alpha\in A$ has at least n^{j-1} descendants $\beta\in L_{(k+1)j}$ and I may choose any possible non-empty subset of these as possible descendants, I get at least

$$(2^{n^{j-1}}-1)^{(1-p^j)(\frac{p^{kj}}{n^k}Ln^{kj-2})-1}$$

eligible descendants $\widetilde{B}\subset\widetilde{A}$. This means that any such $\widetilde{B}$ satisfies

$$\mu(\widetilde{B})\leq(2^{n^{j-1}}-1)^{-(1-p^j)(\frac{p^{kj}}{n^k}Ln^{kj-2})-1}\mu(\widetilde{A}).$$

Applying this recursively I get that a kj-set satisfies

$$\begin{aligned}\mu(\widetilde{A}) &\leq (2^{n^{j-1}}-1)^{-(1-p^j)Ln^{-2}(1+p^jn^{j-1}+\cdots+(p^jn^{j-1})^{k-1})-k}\\ &= (2^{n^{j-1}}-1)^{-(1-p^j)Ln^{-2}\frac{(p^jn^{j-1})^k-1}{p^jn^{j-1}-1}-k}\\ &\leq 2^{-L'(p^jn^{j-1})^k}\end{aligned}$$

where L' is a large enough constant.

The diameter of a kj-set is $\geq u^{kj+1}$. Let $n'=1/u^s<n$. Then

$$\begin{aligned}\frac{\mu(\widetilde{A})}{\psi^s(u^{kj+1})} &\leq \frac{2^{-L'(p^jn^{j-1})^k}}{2^{-(n')^{kj+1}}}\\ &= 2^{(n')^{kj+1}-L'\frac{(p^jn^j)^k}{n^k}}\to 0\end{aligned}$$

Since $p^jn^{j-1}>(n')^j$ by assumption.$\square$

The next order of business is to extend these theorems to more general sets E. For the upper bound, let us suppose that $E\subset F$ where F is the self-similar set given by the maps $(f_1, \ldots, f_m)$ with ratio list $(r_1, \ldots, r_m)$. Let (Ω, ρ) be the corresponding self-similar sequence space. In this situation, it is shown in [Ed1] that there is a surjective Lipschitz map $h : \Omega \to F$. Since a Lipschitz map is continuous and the continuous image of a compact set is compact, h extends naturally to a Lipschitz map $\widetilde{h} : \mathcal{K}(\Omega) \to \mathcal{K}(F)$. Thus the upper bound for $\mathcal{K}(\Omega)$ should hold for $\mathcal{K}(E)$. By composing the map h with another if necessary, it is also clear that F need not be strictly self-similar, but only the Lipschitz image of a self similar set. This is summarized as the following theorem.

Theorem 4.2.3 Let $E \subset F$, where F is the Lipschitz image of a self-similar set with ratio list $(r_1, \ldots, r_m)$ such that $\sum_{i=1}^{m} r_i^{s_0} = 1$. Then there is an $M > 0$ large enough so that $\mathcal{H}^{\varphi_M}(\mathcal{K}(E)) < \infty$.

For the lower bound, suppose that E has an s_0-nested packing. Then we may extract a subset $E' \subset E$ which is bi-Lipschitz equivalent to a self-similar sequence space (Ω, ρ) of finite Hausdorff dimension s_0 by lemma 3.4.1. Again, the bi-Lipschitz map $g : \Omega \to E'$ extends to a bi-Lipschitz map $\tilde{g} : \mathcal{K}(\Omega) \to \mathcal{K}(E')$. Thus I have the following theorem.

Theorem 4.2.4 Suppose that E has an s_0-nested packing. Then for $s < s_0$, I have $\mathcal{H}^{\psi^s}(\mathcal{K}(E)) > 0$.

Next I turn to more concrete examples. Suppose that $X = (x_0, x_1, \ldots, x_\infty)$ is a countable metric space with metric ρ satisfying $\rho(x_n, x_\infty) = a_n \searrow 0$ and $\rho(x_n, x_m) \geq a_n$ for $m < n$. Clearly $\mathcal{H}^{\varphi}(X) = 0$ for every $\varphi \in \Phi$. But the following is also true:

Theorem 4.2.5 Let (X,ρ) be as above and suppose $\varphi \in \Phi$ satisfies $\varphi(a_n) = 2^{-n}$. Then $\dim(\mathcal{K}(X)) \asymp \varphi$.

Proof: A set $T \in \mathcal{K}(X)$ is isolated if and only if $x_\infty \notin T$. Let

$$\mathcal{K}'(X) = \{T \in \mathcal{K}(X) : x_\infty \in T\}.$$

Then $\mathcal{K}(X) \setminus \mathcal{K}'(X)$ is countable so that $\mathcal{H}^\varphi(\mathcal{K}(X) \setminus \mathcal{K}'(X)) = 0$.

Turn now to $\mathcal{K}'(X)$. For fixed $n \in \mathbb{N}$, each set $A \subset \{x_0, \ldots, x_{n-1}\}$ determines a set

$$\widetilde{A}_n = \{T \in \mathcal{K}'(X) : A = T \cap \{x_0, \ldots, x_{n-1}\}\}.$$

Note that if $S, T \in \widetilde{A}_n$, then any point $x_k \in S$ with $k \geq n$ satisfies $\rho(x_k, x_\infty) \leq a_n$. So $\mathrm{dist}(x_k, T) \leq a_n$, since $x_\infty \in T$. Since S and T agree on A, I have that $\mathrm{dist}(x_k, T) \leq a_n$ for every $x_k \in S$ and vice versa. So $\mathrm{diam}(\widetilde{A}_n) \leq a_n$. In fact, $A \cup \{x_\infty\}$ and $A \cup \{x_n, x_\infty\} \in \widetilde{A}_n$, so that $\mathrm{diam}(\widetilde{A}_n) = a_n$. Now there are 2^n such $\widetilde{A}$'s contained in $\{x_0, \ldots, x_{n-1}\}$. So

$$\mathcal{H}^\varphi_{a_n}(\mathcal{K}'(X)) \leq 2^n \varphi(a_n) = 2^n 2^{-n} = 1.$$

So $\mathcal{H}^\varphi(\mathcal{K}'(X)) \leq 1$.

For the lower bound, I will construct a measure μ on $\mathcal{K}'(X)$ recursively. Let $\mu(\mathcal{K}'(X)) = 1$. Fix $m \in \mathbb{N}$ and suppose that μ has been constructed so that $A \subset \{x_0, \ldots, x_{n-1}\}$ implies $\mu(\widetilde{A}_n) = 2^{-n}$ for every $n \leq m$. Note that if $A \subset \{x_0, \ldots, x_{m-1}\}$, then

$$\widetilde{A}_m = \{T \in \widetilde{A}_m : x_m \in T\} \cup \{S \in \widetilde{A}_m : x_m \notin S\}.$$

Divide $\mu(\widetilde{A}_m)$ evenly between these two sets. In this way μ is constructed so that $\mu(\widetilde{A}_n) = 2^{-n}$ for any $n \in \mathbb{N}$ and $A \subset \{x_0, \ldots, x_{n-1}\}$.

Now suppose that $\widetilde{B} \subset \mathcal{K}'(X)$ satisfies $a_{n+1} < \operatorname{diam}(\widetilde{B}) \leq a_n$. Let $T \in \widetilde{B}$ and let $A = T \cap \{x_0, \ldots, x_{n-1}\}$. Then $\widetilde{B} \subset \widetilde{A}_n$, so

$$\mu(\widetilde{B}) \leq 2^{-n} = 2 \cdot 2^{-(n+1)} = 2\varphi(a_{n+1}) < 2\varphi(\operatorname{diam}(\widetilde{B})).$$

Thus $\mathcal{H}^{\varphi}(\mathcal{K}'(X)) \geq 1/2$, by 3.3.2.□

4.3 Entropy and Hausdorff Dimensions of the Space $\mathcal{C}(\mathbb{R}^d)$

In this section I investigate the entropy and Hausdorff dimensions of the set of compact convex subsets of $\mathbb{R}^d$, denoted $\mathcal{C}(\mathbb{R}^d)$, endowed with the Hausdorff metric $\widetilde{\rho}$. This is a closed subspace of $\mathcal{K}(\mathbb{R}^d)$ and so is a complete separable metric space. A good reference for general facts about $\mathcal{C}(\mathbb{R}^d)$ is [Sch].

The case $d = 1$ is easy. Any closed convex subset of $\mathbb{R}$ is a closed interval which is uniquely determined by its endpoints. $\mathcal{C}(\mathbb{R})$ is therefore two dimensional. Results for $d \geq 2$ rest on the independent work in [Bro] and [Dud] which essentially compute the entropy indices. In particular, theorem 5 in [Bro] states the following:

Theorem 4.3.1 Let $T_d = B_1(0)$ be the open ball of radius 1 about the origin in $\mathbb{R}^d$. Then for $0 < \varepsilon \leq 1/(10^{12}(d-1))$,

$$2^{a_d(1/\varepsilon)^{\frac{d-1}{2}}} \leq N_\varepsilon(\mathcal{C}(T_d)) \leq 2^{b_d(1/\varepsilon)^{\frac{d-1}{2}}},$$

where $a_d < b_d$ are positive constants depending on d.

In my notation, this implies that if $\varphi_a(t) = 2^{-a(1/t)^{\frac{d-1}{2}}}$, then

$$\varphi_{a_d} \preceq \delta(\mathcal{C}(T_d)) \preceq \Delta(\mathcal{C}(T_d)) \preceq \varphi_{b_d}.$$

My first contribution is to extend this to the entropy dimensions.

Theorem 4.3.2 Let $\varphi_a(t) = 2^{-a(1/t)^{\frac{d-1}{2}}}$ and suppose that $\varphi \prec \varphi_a \prec \psi$ for every $a > 0$. Then for every $r > 0$,

$$\varphi \prec \widehat{\delta}(\mathcal{C}(B_r(0))) \preceq \widehat{\Delta}(\mathcal{C}(\mathbb{R}^d)) \prec \psi.$$

Proof: Let T_d denote the open unit ball in R^d. If $f : T_d \to B_r(0)$ is a similarity with ratio $r > 0$, then $\widetilde{f} : \mathcal{C}(T_d) \to \mathcal{C}(B_r(0))$ given by $\widetilde{f}(E) = f(E)$ is also a similarity with ratio r. It then follows from theorem 4.3.1 that

$$2^{r^{\frac{d-1}{2}} a_d (1/\varepsilon)^{\frac{d-1}{2}}} \leq N_\varepsilon(\mathcal{C}(B_r(0))) \leq 2^{r^{\frac{d-1}{2}} b_d (1/\varepsilon)^{\frac{d-1}{2}}}.$$

So

$$\varphi_a \prec \delta(\mathcal{C}(B_r(0))) \preceq \Delta(\mathcal{C}(B_r(0))) \prec \varphi_b, \tag{4.2}$$

whenever $a < r^{\frac{d-1}{2}} a_d \leq r^{\frac{d-1}{2}} b_d < b$.

For the lower bound I will apply a Baire category argument to the set $\mathcal{C}(B_r(0))$ which is not complete. $\mathcal{C}(B_r(0))$ is easily seen to be open in $\mathcal{C}(\mathbb{R}^d)$ and is therefore topologically complete by theorem 12. 1 in [Oxt]. This means that it may be remetrized by means of a complete metric and so the Baire category theorem still holds.

Fix $r > 0$ and suppose $\mathcal{C}(B_r(0)) = \cup_n \widetilde{C}_n$, where each $\widetilde{C}_n$ is closed. By Baire category, there are $n \in \mathbb{N}$, $\varepsilon > 0$, and $E \in \mathcal{C}(B_r(0))$ such that $\widetilde{B}_\varepsilon(E) \subset \widetilde{C}_n$. I claim that $\widetilde{B}_\varepsilon(E) \supset \{E\} + \mathcal{C}(B_\varepsilon(0))$ where $+$ stands for set addition ($E + A = \{x + y : x \in E, y \in A\}$). This is because set addition is an isometry in $\mathcal{C}(\mathbb{R}^d)$ (see [Sch] page 59). So if $A \subset B_\varepsilon(0)$, then $\widetilde{\rho}(E, E + A) < \varepsilon$. We also see that $\{E\} + \mathcal{C}(B_\varepsilon(0))$ is isometric to $\mathcal{C}(B_\varepsilon(0))$ which satisfies $\varphi \prec \varphi_a \prec \delta(\mathcal{C}(B_\varepsilon(0)))$ for some $a > 0$ by equation 4.2. Thus $\varphi \prec \delta(\widetilde{C}_n)$ and $\varphi \prec \widehat{\delta}(\mathcal{C}(B_r(0)))$.

For the upper bound, write $\mathcal{C}(\mathbb{R}^d) = \cup_{n\in\mathbb{N}}\mathcal{C}(B_n(0))$. Then by equation 4.2, for every $n \in \mathbb{N}$, there is a $b_n > 0$ so that $\Delta(\mathcal{C}(\mathcal{B}_n(0))) \prec \varphi_{b_n} \prec \psi$. So $\widehat{\Delta}(\mathcal{C}(\mathbb{R}^d)) \prec \psi$.$\Box$

I now turn to the computation of the Hausdorff dimension. The upper bound given above for the entropy dimensions holds for the Hausdorff dimension as well by the comparison theorems 2.1.1 and 2.2.2. The lower bound for the Hausdorff dimension is somewhat weaker and the proof is considerably more difficult. A good reference for the Riesz representation theorem and Alaoglu's theorem as used in the next proof is [Con].

Theorem 4.3.3 Let $\psi^s(t) = 2^{-(1/t)^s}$ and suppose that $s < \frac{d-1}{2}$. Then $\dim(\mathcal{C}(\mathbb{R}^d)) \succ \psi^s$.

Proof: Let $T = B_1(0) \subset \mathbb{R}^d$ be the closed unit ball. I will need to obtain a measure μ on $\mathcal{C}(T)$ as the weak-$*$ limit of some discrete measures. Let $S^{d-1} = \partial T^d$ be the unit sphere. Take a $d-1$ dimensional cube of side length < 1 and project it orthogonally onto S^{d-1}. Denote this region of S^{d-1} so obtained by C. Then C is the bi-Lipschitz image of a self-similar $d-1$ dimensional set and so has a $2s$-nested packing since $2s < d-1$. Denote this $2s$-nested packing by $\{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha\in\Omega^*}$ where $\Omega = \{1, \ldots, m\}^{\mathbb{N}}$.

Then I may extract a packing $\mathcal{B} \subset \{B_{c\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha\in\Omega^*}$ such that for all $n \in \mathbb{N}$,

$$\#(\{\alpha \in \Omega^n : B_{c\varepsilon^n}(x_\alpha) \in \mathcal{B}\}) = (m-1)^{n-1}$$

by lemma 3.4.2. For $n \in \mathbb{N}$, let

$$\mathcal{B}_n = \{x_\alpha \in C : B_{c\varepsilon^{|\alpha|}}(x_\alpha) \in \mathcal{B} \text{ and } |\alpha| \leq n\}.$$

Let $x_0 \in S^{d-1}$ be a point opposite C (i.e. $x_0 \in -C$). Each subset $F \subset \mathcal{B}_n$ determines a convex subset $\widetilde{F} \subset T^d$ given by $\widetilde{F} = \mathrm{conv}(F \cup \{x_0\})$, where conv denotes the closed convex hull. Let G_n denote the set of all such convex sets generated by $\mathcal{B}_n$. Then

$$\#(\mathcal{B}_n) = 1 + (m-1) + (m-1)^2 + \cdots + (m-1)^{n-1} = \frac{(m-1)^n - 1}{m-2}$$

and $\#(G_n) = 2^{\frac{(m-1)^n-1}{m-2}}$. The elements of G_n are all distinct. In fact, there is a convenient lower bound on their separation. If $x_\alpha \in E \setminus F$, where $E, F \subset \mathcal{B}_n$ and $|\alpha| = k$, then any point of F is separated from x_α by at least $c\varepsilon^k$. The argument used in [Bro] in the proof of 4.3.1 then shows that there is an $a > 0$ independent of k such that $\widetilde{\rho}(\widetilde{E}, \widetilde{F}) \geq a\varepsilon^{2k}$. Let μ_n be normalized counting measure on G_n. Each μ_n is a continuous linear functional on $C_0(\mathcal{C}(T^d))$ of norm 1 by the Riesz representation theorem. The unit ball in $C_0(\mathcal{C}(T^d))^*$ is weak-$*$ compact by Alaoglu's theorem, so $(\mu_n)_{n=1}^\infty$ has a weak-$*$ cluster point, say μ.

The measure μ above will be used in a density argument, so I need to be able to compare the μ-measure of a set with its diameter. So fix $n \in \mathbb{N}$ and suppose $\widetilde{A} \subset \mathcal{C}(T^d)$ is open and satisfies

$$a\varepsilon^{2(n+1)} \leq \mathrm{diam}(\widetilde{A}) < a\varepsilon^{2n}.$$

To estimate $\mu(\widetilde{A})$, it suffices to estimate $\mu_j(\widetilde{A})$ for large j as μ is a weak-$*$ cluster point of $(\mu_j)_j$. This in turn depends on $\#(\widetilde{A} \cap G_j)$. The restriction on $\mathrm{diam}(\widetilde{A})$ guarantees that $\widetilde{A}$ cannot contain sets $E, F \in G_j$ which disagree at some point $x_\alpha \in \mathcal{B}_n$. On the other hand, $\widetilde{A}$ is potentially large enough to accommodate all possible choices for $x_\alpha \in \mathcal{B}_j \setminus \mathcal{B}_n$. Note

$$\begin{aligned} \#(\mathcal{B}_j \setminus \mathcal{B}_n) &= (m-1)^{j-1} + (m-1)^{j-2} + \cdots + (m-1)^n \\ &= \frac{(m-1)^j - (m-1)^n}{m-2}, \end{aligned}$$

so $\#(\widetilde{A} \cap G_j) \leq 2^{\frac{(m-1)^j - (m-1)^n}{m-2}}$. An element $E \in G_j$ has $\mu_j(\{E\}) = 2^{\frac{1-(m-1)^j}{m-2}}$. So

$$\mu_j(\widetilde{A}) \leq 2^{\frac{(m-1)^j - (m-1)^n}{m-2}} 2^{\frac{1-(m-1)^j}{m-2}} = 2^{\frac{1-(m-1)^n}{m-2}},$$

for every $j > n$. So $\mu(\widetilde{A}) \leq 2^{\frac{1-(m-1)^n}{m-2}}$.

Now $\psi^s(\mathrm{diam}(\widetilde{A})) \geq 2^{-(1/a\varepsilon^{2(n+1)})^s} = 2^{-a'(1/\varepsilon^{2s})^n}$, where $a' > 0$ is a constant. So

$$\frac{\mu(\widetilde{A})}{\psi^s(\mathrm{diam}(\widetilde{A}))} \leq 2^{\frac{1-(m-1)^n}{m-2}} 2^{a'(1/\varepsilon^{2s})^n} \to 0$$

as $n \to \infty$, since $1/\varepsilon^{2s} < m-1$ by the definition of a $2s$-nested packing. This yields the result by lemma 3.3.4.$\square$

CHAPTER V

Function Spaces

In [KolTi] the entropy indices are investigated for various compact sets of functions. Of central importance there are sets of functions of some prescribed degree of smoothness. Roughly speaking, if $E \subset \mathbb{R}^d$ has $\dim(E) \sim t^s$ and F is the set of functions defined on E and smooth of order $q > 0$, then $\dim(F) \sim 2^{-(1/t)^{s/q}}$. Thus as E increases in dimension, so does F. While an increase in the order of smoothness of the functions in F, decreases the dimension of F. In this chapter I extend these theorems to some other notions of dimension.

5.1 Entropy Dimensions of the Space of Smooth Functions

In order to make the above ideas precise, I will first need to carefully define the set of functions under consideration. Let $c > 0$, $d \in \mathbb{N}$, $E \subset \mathbb{R}$ be compact, and let $q = p + \lambda$ where $p \in \mathbb{N}$ and $\lambda \in (0, 1]$. Denote the set of real valued bounded continuous functions defined on $\mathbb{R}^d$ by $C(\mathbb{R}^d)$. Similarly, $C(E)$ denotes the set of real valued continuous (and necessarily bounded) functions on E. If $f \in C(\mathbb{R}^d)$, then $f|_E$ will denote the restriction of f to E. The uniform norm is denoted by $\| \cdot \|$

and will be applied to functions in both $C(\mathbb{R}^d)$ and $C(E)$. I want to define a set $F(c,q,E) \subset C(E)$ which are to be thought of as smooth of order q and bound by c. First I need to define a similar set $F(c,q,\mathbb{R}^d) \subset C(\mathbb{R}^d)$.

To define $F(c,q,\mathbb{R}^d)$ a convenient notation for the partial derivatives of a function $f \in C(\mathbb{R}^d)$ will be needed. A multi-index α is a vector in $\mathbb{N}^d$. This is not to be confused with the notion of an initial segment. If $\alpha = (\alpha_1, \ldots, \alpha_d)$ is a multi-index, then write $|\alpha| = \alpha_1 + \cdots + \alpha_d$. If $f \in C(\mathbb{R}^d)$ is $|\alpha|$-times differentiable, then $D^\alpha f(x)$ denotes the partial derivative $\frac{\partial^{|\alpha|} f(x)}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_d} x_d}$ where $x = (x_1, \ldots, x_n)$.

$F(c,q,\mathbb{R}^d)$ denotes the set of all functions $f \in C(\mathbb{R}^d)$ such that:

1. $D^\alpha f$ exists and satisfies $\|D^\alpha f\| \leq c$ for all multi-indices α with $|\alpha| \leq p$.

2. $|D^\alpha f(x) - D^\alpha f(y)| \leq c|x-y|^\lambda$ for all $x, y \in \mathbb{R}^d$ and multi-indices α with $|\alpha| = p$.

Note that condition 1 above includes the case $|\alpha| = 0$ so that f itself satisfies $\|f\| \leq c$. The set $F(c,q,E)$ is defined by restricting functions from $F(c,q,\mathbb{R}^d)$ to E. So write $g \in F(c,q,E)$ if $g = f|_E$, where $f \in F(c,q,\mathbb{R}^d)$. The following theorem is a slight modification of theorem 15 in [KolTi].

Theorem 5.1.1 Fix $c, q, s_1, s_2 > 0$ and suppose that $E \subset \mathbb{R}^d$ is a compact set satisfying

$$t^{s_1} \preceq \delta(E) \preceq \Delta(E) \preceq t^{s_2}.$$

Then there are $a, b > 0$ such that

$$2^{-a(1/t)^{s_1/q}} \preceq \delta(F(c,q,E)) \preceq \Delta(F(c,q,E)) \preceq 2^{-b(1/t)^{s_2/q}}.$$

I should point out that the definition of $F(c,q,E)$ in [KolTi] is somewhat more abstract. However, a close reading of the proof there yields this theorem for the definition of $F(c,q,E)$ given here as well.

The goal in this section is to extend theorem 5.1.1 to the entropy dimensions. As $\widehat{\Delta} \preceq \Delta$ in general, the upper bound is not a difficulty. To extend the lower bound to $\widehat{\delta}$ will require a Baire category argument. This in turn requires that $F(c,q,E)$ is uniformly closed in $C(E)$. I have not been able to find this in the literature and so prove it now. This will require several lemmas. The first goal will be to show that $F(c,q,\mathbb{R}^d)$ is closed in $C(\mathbb{R}^d)$ with respect to uniform convergence on compact subsets of $\mathbb{R}^d$. This mode of convergence will henceforth be abbreviated convergence UCS. The notion of equicontinuity will play a crucial role. A family F of real valued functions defined on a metric space (X,ρ) is said to be *equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $f \in F$ and $\rho(x,y) < \delta$. Note that a subset of an equicontinuous family of functions is again equicontinuous.

Lemma 5.1.1 Let $c > 0$, $q \in (0,1]$, and $f_n \in F(c,q,\mathbb{R}^d)$ for every $n \in \mathbb{N}$ such that $f_n \to f$ as $n \to \infty$ pointwise. Then $f \in F(c,q,\mathbb{R}^d)$.

Proof: Clearly $\|f\| \le c$. To establish the Hölder condition, let $x, y \in \mathbb{R}^d$, $\varepsilon > 0$, and choose $n_0 \in \mathbb{N}$ such that $n > n_0$ implies that $|f_n(x) - f(x)| < \varepsilon/2$ and $|f_n(y) - f(y)| < \varepsilon/2$. Then for $n > n_0$,

$$\begin{aligned} |f(x) - f(y)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\le \frac{\varepsilon}{2} + c|x-y|^q + \frac{\varepsilon}{2} = c|x-y|^q + \varepsilon. \end{aligned}$$

Thus $|f(x) - f(y)| \leq c|x-y|^q$, as $\varepsilon > 0$ is arbitrary.$\square$

Lemma 5.1.2 Let $c > 0$ and let $q = p + \lambda$ where $p \in \mathbb{N}$ and $\lambda \in (0,1]$. If α is a multi-index with $|\alpha| \leq p$, then the family $\{D^\alpha f : f \in F(c, q, \mathbb{R}^d)\}$ is equicontinuous.

Proof: If $|\alpha| = p$, then I may apply the Hölder condition. So let $\varepsilon > 0$ and choose $0 < \delta < (\varepsilon/c)^{1/\lambda}$. Then $|x - y| < \delta$ implies

$$|D^\alpha f(x) - D^\alpha f(y)| \leq c|x-y|^\lambda < c\left((\varepsilon/c)^\lambda\right)^\lambda = \varepsilon.$$

If $|\alpha| < p$, then by the mean value theorem and the bound on $D^\beta(D^\alpha f)$ where $|\beta| = 1$, we have that $D^\alpha f$ satisfies the Lipschitz condition

$$|D^\alpha f(x) - D^\alpha f(y)| \leq \|\mathrm{grad}(D^\alpha f)\| \cdot |x-y| \leq dc|x-y|.$$

Thus the same argument applies.$\square$

The following lemma is commonly known as the Arzelá-Ascoli Theorem and appears in [Fol] as theorem 4.44 (b).

Lemma 5.1.3 If $(f_n)_n$ is an equicontinuous pointwise bounded sequence of real valued functions defined of $\mathbb{R}^d$, then there is a subsequence $(f_{n_j})_j$ and $f \in C(\mathbb{R}^d)$ such that $f_{n_j} \to f$ UCS.

The final lemma appears as theorem 4.56 in [Str].

Lemma 5.1.4 Let $I \subset \mathbb{R}$ be a compact interval and let $(f_n)_n$ be a sequence of real valued differentiable functions on I. Suppose that $f_n' \to g$ uniformly on I and that for some $c \in I$, $f_n(c)$ converges. Then there is a differentiable function f defined on I such that $f_n \to f$ uniformly on I and $f' = g$.

Corollary 5.1.1 Suppose that $f_n \to f$ pointwise on $\mathbb{R}$ and $f_n' \to g$ UCS. Then f is differentiable and $f' = g$.

Theorem 5.1.2 $F(c, q, \mathbb{R}^d)$ is closed in $C(\mathbb{R}^d)$ with respect to the topology of uniform convergence on compact subsets of $\mathbb{R}^d$.

Proof: The case $0 < q \leq 1$ is contained in lemma 5.1.1. So suppose that $q > 1$. For every $n \in \mathbb{N}$, let $f_n \in F(c, q, \mathbb{R}^d)$ and suppose that $f_n \to f$ UCS. I want to show that $f \in F(c, q, \mathbb{R}^d)$.

I will recursively define a sequence $n_j \nearrow \infty$ such that $D^\alpha f_{n_j}$ converges UCS as $j \to \infty$ for every multi-index α with $|\alpha| \leq p$. Let M_p be the number of multi-indices of length $\leq p$ and let $(\alpha_i)_{i=1}^{M_p}$ be a list of those multi-indices with $|\alpha_1| = 0$. Let $j(1) \nearrow \infty$ be just the sequence of natural numbers. Then $D^{\alpha_1} f_{n_{j(1)}} = f_{n_{j(1)}} \to f$ UCS as $j(1) \to \infty$. Suppose that for $k - 1 < M_p$, the sequence $j(k-1) \nearrow \infty$ has been defined so that $D^{\alpha_i} f_{n_{j(k-1)}}$ converges UCS to say g_{α_i} as $j(k-1) \to \infty$ for every $i = 1, \ldots, k-1$. By lemma 5.1.2 the set $\{D^{\alpha_k} f_{n_{j(k-1)}}\}_{j(k-1)}$ is equicontinuous. It is also uniformly bounded by c and so by lemma 5.1.3 there is a subsequence $j(k)$ of $j(k-1)$ so that $(D^{\alpha_k} f_{n_{j(k)}})_{j(k)}$ converges UCS as $j(k) \to \infty$ to say g_{α_k}. This process terminates when $j(M_p)$ is reached to generate the final sequence $(n_j)_j$ such that $D^\alpha f_{n_j} \to g_\alpha$ UCS as $j \to \infty$ for all multi-indices α with $|\alpha| \leq p$.

Next suppose that α is a multi-index with $|\alpha| = 1$. Then viewing $(f_{n_j})_j$ as a sequence of functions of the single variable with respect to which D^α differentiates, we see that corollary 5.1.1 applies to the sequences $f_{n_j} \to f$ UCS and $D^\alpha f_{n_j} \to g_\alpha$ UCS. Thus f is differentiable and $D^\alpha f = g_\alpha$. Clearly $\|D^\alpha f\| \leq c$ since $\|D^\alpha f_{n_j}\| \leq c$ for every j. This same argument may be recursively applied to obtain $D^\alpha f = g_\alpha$ and $\|D^\alpha f\| \leq c$ for all multi-indices α with $|\alpha| \leq p$.

Finally, for $|\alpha| = p$, $D^\alpha f$ satisfies the necessary Hölder condition by lemma 5.1.1.□

Corollary 5.1.2 $F(c, q, E)$ is compact (and therefore closed) with respect to the uniform norm.

Proof: Let $g_n \in F(c, q, E)$ for every $n \in \mathbb{N}$. So $g_n = f_n|_E$ where $f_n \in F(c, q, \mathbb{R}^d)$. The sequence $(f_n)_n$ is uniformly bounded by c. By lemma 5.1.2, $(f_n)_n$ forms an equicontinuous family. Thus by lemma 5.1.3, there is a function $f \in C(\mathbb{R}^d)$ and a subsequence $(f_{n_j})_j$ of $(f_n)_n$ such that $f_{n_j} \to f$ UCS as $j \to \infty$. By theorem 5.1.2, $f \in F(c, q, \mathbb{R}^d)$. Thus the function $g = f|_E$ satisfies $g \in F(c, q, E)$ and $g_{n_j} \to g$ uniformly on E.□

One more lemma is necessary for the main theorem of the section.

Lemma 5.1.5 Let $c_1, c_2, q, \gamma > 0$, $f \in F(c_1, q, \mathbb{R}^d)$, and $g \in F(c_2, q, \mathbb{R}^d)$. Then $f + g \in F(c_1 + c_2, q, \mathbb{R}^d)$ and $\gamma f \in F(\gamma c_1, q, \mathbb{R}^d)$.

Proof: If α is a multi-index with $|\alpha| \leq p$, then $D^\alpha(f + g) = D^\alpha f + D^\alpha g$ and

$$\|D^\alpha(f + g)\| \leq \|D^\alpha f\| + \|D^\alpha g\| \leq c_1 + c_2.$$

If $|\alpha| = p$ and $x, y \in \mathbb{R}^d$, then

$$\begin{aligned} |D^\alpha(f+g)(x) - D^\alpha(f+g)(y)| &\leq |D^\alpha f(x) - D^\alpha f(y)| + |D^\alpha g(x) - D^\alpha g(y)| \\ &\leq c_1|x-y|^\lambda + c_2|x-y|^\lambda = (c_1+c_2)|x-y|^\lambda. \end{aligned}$$

Similarly, $\|D^\alpha(\gamma f)\| \leq \gamma c_1$ and

$$\|D^\alpha(\gamma f(x)) - D^\alpha(\gamma f(y))| \leq \gamma c_1 |x-y|^\gamma. \square$$

Corollary 5.1.3 Let $c_1, c_2, q, \gamma > 0$, $E \subset \mathbb{R}^d$ be compact, $f \in F(c_1, q, E)$, and $g \in F(c_2, q, E)$. Then $f + g \in F(c_1 + c_2, q, E)$ and $\gamma f \in F(\gamma c_1, q, E)$.

Finally, the following theorem gives a lower bound found for $\widehat{\delta}(F(c, q, E))$.

Theorem 5.1.3 Let $s, q, c > 0$ be fixed and suppose that $\varphi \prec 2^{-a(1/t)^{s/q}}$ for every $a > 0$. If the compact set $E \subset \mathbb{R}^d$ satisfies $\delta(E) \succeq t^s$ then $\widehat{\delta}(F(c, q, E)) \succ \varphi$.

Proof: Suppose that $F(c, q, E) = \cup_n F_n$ where each F_n is closed. By Baire category there is an $n \in \mathbb{N}$ an $f \in F_n$, and an $r > 0$ such that $B_r(f) \subset F_n$. I may assume, in fact, that $f \in F(c', q, E)$ where $0 < c' < c$. Otherwise, just replace f with γf where $\gamma < 1$ is close enough to 1 so that $\gamma f \in B_r(f)$. Then $\gamma f \in F(\gamma c, q, E)$ and there is an $r' > 0$ such that $B_{r'}(\gamma f) \subset B_r(f)$.

Consider now $g \in F(\min(r, c - c'), q, E)$. By corollary 5.1.3, $g + f \in F(c, q, E)$. Thus

$$g \in B_r(f) - \{f\} = \{h - f : h \in B_r(f)\}$$

since we may take $h = g + f$. $B_r(f) - \{f\}$ is clearly isometric to $B_r(f)$ and we see that

$$B_r(f) - \{f\} \supset F(\min(r, c - c'), q, E).$$

Thus by theorem 5.1.1, there is an $a > 0$ such that $\delta(F_n) \succ \delta(B_r(f)) \succ 2^{-a(1/t)^{s/q}} \succ \varphi$ and $\widehat{\delta}(F(c, q, E)) \succ \varphi$.□

It is interesting to note that the bound on $\widehat{\delta}(F(c, q, E))$ is expressed in terms of $\delta(E)$ rather than $\widehat{\delta}(E)$. thus a set small with respect to $\widehat{\delta}$ (but large with respect to δ) can lead to a set of functions large with respect to $\widehat{\delta}$. For example, the set $E = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\} \subset \mathbb{R}$ satisfies $\delta(E) \preceq \Delta(E) \prec \varphi$ for every $\varphi \in \Phi$ since E is countable. E also satisfies $\delta(E) \asymp \Delta(E) \asymp t^{1/2}$ ([Fal2] example 3.5). Thus for $0 < s < 1/2$ and $q > 0$ we have $\widehat{\delta}(F(c, q, E)) \succ 2^{-(1/t)^{s/q}}$.

5.2 Hausdorff Dimension of the Space of Smooth Functions

In this section, I investigate the Hausdorff dimension of $F(c, q, E)$. The upper bound given in theorem 5.1.1 holds for the Hausdorff dimension as well, so I concentrate here on the lower bound. As with the space $\mathcal{K}(E)$, I need to make stronger assumptions on the set E.

Theorem 5.2.1 Fix $s, q, c > 0$ and suppose the compact set E has an s-nested packing. Then $\dim(F(c, q, E)) \succ \varphi_{s/q}(t) = 2^{-(1/t)^{s/q}}$.

Proof: Let

$$g(x) = \begin{cases} a \prod_{i=1}^{d} (1 + x_i)^q (1 - x_i)^q & \text{if all } |x_i| < 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $x = (x_0, \ldots, x_d) \in \mathbb{R}^d$ and $a > 0$ is a constant. Note that if a is chosen small enough, then $g \in F(c,q,E)$. Note also that for $b \in (0,1)$ we have $b^q g(x/b) \in F(c,q,E)$.

Now E has an s-nested packing say $\{B_{c_0\varepsilon^{|\alpha|}}(x_\alpha)\}_{\alpha\in\Omega^*}$. Extract from this a packing $\mathcal{B}$ satisfying

$$\#(\{\alpha \in \Omega^n : B_{c_0\varepsilon^n}(x_\alpha) \in \mathcal{B}\}) = (m-1)^{n-1},$$

as guaranteed by lemma 3.4.2. Recall that m, c_0, ε, and s are all fixed and satisfy $m > (1/\varepsilon)^s + 1$. Define $\mathcal{B}_n = \{B_{c_0\varepsilon^n}(x_\alpha) \in \mathcal{B} : \alpha \in \Omega^n\}$. I'll now define some discrete sets and argue in a way very similar to theorem 4.3.3. Let G_n be the set of all those functions $f \in F(c,q,E)$ of the form:

$$f(x) = \begin{cases} \pm(\frac{c_0\varepsilon}{\sqrt{d}})^q g(\frac{\sqrt{d}(x-x_\alpha)}{c_0\varepsilon}) & \text{for } x \in B_{c_0\varepsilon}(x_\alpha) \in \mathcal{B}_1 \\ \vdots & \vdots \\ \pm(\frac{c_0\varepsilon^n}{\sqrt{d}})^q g(\frac{\sqrt{d}(x-x_\alpha)}{c_0\varepsilon^n}) & \text{for } x \in B_{c_0\varepsilon^n}(x_\alpha) \in \mathcal{B}_n \\ 0 & \text{for all other } x. \end{cases}$$

Now $\#(\mathcal{B}_k) = (m-1)^{k-1}$ and for each $B \in \cup_{k=1}^n \mathcal{B}_k$ there are two choices to form an $f \in G_n$ induced by the $\pm$ above. So

$$\begin{aligned} \#(G_n) &= 2 \cdot 2^{m-1} \cdots 2^{(m-1)^{n-1}} \\ &= 2^{1+(m-1)+\cdots+(m-1)^{n-1}} = 2^{\frac{(m-1)^n - 1}{m-2}}. \end{aligned}$$

As in the proof of theorem 4.3.3, let μ_n be normalized counting measure on G_n and let μ be a weak-$*$ cluster point of $\{\mu_n\}$. Let $A \subset F(c,q,E)$ be open and satisfy

$$2a(\frac{c_0}{\sqrt{d}})^q \varepsilon^{(n+1)q} \leq \operatorname{diam}(A) < 2a(\frac{c_0}{\sqrt{d}})^q \varepsilon^{nq}.$$

I want an upper bound on $\mu(A)$. For this it suffices to have a bound on $\mu_j(A)$ for large j, which I get by estimating $\#(A\cap G_j)$. Note that if two functions $f_1, f_2 \in G_j$ disagree on some $B \in \mathcal{B}_k$, then they will disagree by at least $2a(\frac{c_0}{\sqrt{d}})^q \varepsilon^{nq}$. So A is big enough to allow functions to do what they like on any $B \in \mathcal{B}_k$ if $k = n+1, \ldots, j$, but functions in $A \cap G_n$ are restricted to just one choice on any $B \in \mathcal{B}_k$ for $k = 1, \ldots, n$. So for $j > n$,

$$\begin{aligned}
\#(A\cap G_j) &\leq 2^{(m-1)^{j-1}} \cdot 2^{(m-1)^{j-2}} \cdots 2^{(m-1)^n} \\
&= 2^{(m-1)^n((m-1)^{j-n-1}+(m-1)^{j-n-2}+\cdots+1)} \\
&= 2^{(m-1)^n\left(\frac{(m-1)^{j-n}-1}{m-2}\right)} = 2^{\frac{(m-1)^j-(m-1)^n}{m-2}}.
\end{aligned}$$

Any $f \in G_j$ satisfies $\mu_j(\{f\}) = 2^{\frac{1-(m-1)^j}{m-2}}$, so

$$\mu_j(A) \leq 2^{\frac{(m-1)^j-(m-1)^n}{m-2}} 2^{\frac{1-(m-1)^j}{m-2}} = 2^{\frac{1-(m-1)^n}{m-2}}$$

and $\mu(A) \leq 2^{\frac{1-(m-1)^n}{m-2}}$.

Now $\operatorname{diam}(A) \geq 2a(\frac{c_0}{\sqrt{d}})^q \varepsilon^{(n+1)q}$, so

$$\varphi_{s/q}(\operatorname{diam}(A)) \geq 2^{-(2a(\frac{c_0}{\sqrt{d}})^q\varepsilon^{(n+1)q})^{-s/q}} = 2^{-c'(1/\varepsilon^s)^n},$$

where c' is constant. Thus

$$\frac{\mu(A)}{\varphi_{s/q}(\operatorname{diam}(A))} \leq 2^{\frac{1-(m-1)^n}{m-2}} 2^{c'(1/\varepsilon^s)^n} \to 0$$

as $n \to \infty$, since $m - 1 > 1/\varepsilon^s$. This implies the result by lemma 3.3.4.$\square$

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