A Stochastic Cellular Automaton for Three-Coloring Penrose Tiles

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Abstract

We present a three state, stochastic cellular automaton that runs on Penrose tilings and seems to evolve to a three-colored equilibrium.

1 Introduction

In 1973 and 1974, Roger Penrose discovered three sets of polygons each of which tiles the plane aperiodically and (if certain matching conditions are enforced) only aperiodically. Later, John H. Conway asked if such tilings can be three-colored, where adjacent tiles are to receive different colors. This question has been answered affirmatively for two types of Penrose tilings, but appears to be open for the remaining type. In this paper, we present an algorithm that seems to three-color finite parts of Penrose tilings of all types. The algorithm works by running a particular three-state, stochastic cellular automaton on a given Penrose tiling. The cellular automaton is chosen so that three-colorings are stable and it seems to generally evolve to such an equilibrium.

2 Penrose tilings

There are three types of Penrose tilings: tilings by kites and darts, tilings by rhombs, and tilings by pentacles. We describe them briefly here. More detailed references are [2] and [3].

2.1 Tilings by kites and darts

Figure 1 shows the kite and dart. The sides have length either 1 or τ, the golden ratio, and the angles are all integer multiples of $\frac{\pi}{5}$. The filled and unfilled disks at the vertices are used to enforce a matching condition. When tiling the plane with kites and darts, we demand that filled disks meet filled disks and unfilled disks meet unfilled disks. This matching condition guarantees that any tiling by kites and darts will be aperiodic, i.e. no translation of the tiling maps each tile to another tile. Figure 2 shows part of such a tiling.

2.2 Tilings by rhombs

Figure 3 shows the fat and skinny rhombs. The sides all have length 1 and the angles are all integer multiples of $\frac{\pi}{5}$. The matching condition is slightly more complicated. We demand that filled disks meet filled disks, unfilled disks meet unfilled disks, and oriented edges meet with the correct orientation. Again, this matching condition guarantees that any tiling by rhombs will be aperiodic. Figure 4 shows part of such a tiling.

2.3 Tilings by pentacles

Figure 5 shows the pentacles. As with the rhombs, all of the sides have length one and the angles are all integer multiples of $\frac{\pi}{5}$. The labels indicate a matching condition, which again assures aperiodicity. The edges labeled 0 must fit against edges labeled $\bar{0}$, 1 against $\bar{1}$, and 2 against $\bar{2}$. Note that the three pentagons are congruent, but have different matching conditions. A portion of a tiling by pentacles is shown in figure 6.

3 Coloring the tiles

A tiling is called *three-colorable* if we may assign one of three distinct colors to each tile such that adjacent tiles have different colors. Tiles are said to be adjacent if their intersection is a line segment. Figures 7, 8, and 9 show three-colored tilings by kites and darts, rhombs, and pentacles respectively. Sibley and Wagon [5] proved that tilings by rhombs are three-colorable and Babilon [1] proved that tilings by kites and darts are three-colorable. The equivalent question for the pentacles seems to be open.

Our pictures are the final stage of a three-state, stochastic cellular automaton that can run on any tiling. The cellular automaton works as follows. First, assign one of three possible colors to each tile randomly. Then, allow the cellular automaton to evolve according to the following set of rules:

- If the value of a cell (or tile) equals the value of a bordering cell that is closer to the origin (as measured by some arbitrary point chosen within each tile), then with 90% probability, the cell changes value randomly to one of the other two colors.
- If the value of a cell does not equal the value of a bordering cell that is closer to the origin, but does equal the value of a cell farther away from the origin, then with 10% probability, the cell changes value.
- If the value of the cell does not equal the value of any bordering cell, the cell does not change value.

Note that three-colorings are stable under these rules. The hope is that three-colorings are attractive equilibria. Figure 10 demonstrates the algorithm on a small piece of a kite and dart tiling. The origin is located at the lower left corner of the triangle. The probabilities are chosen so that tiles close to the origin are generally colored correctly before tiles farther away from the origin. The values of 90% and 10% were chosen experimentally. Larger dynamic images are available on the author's web page:
http://www.unca.edu/~mcmcclur/mathematicaGraphics/PenroseColoring/

4 Related coloring schemes

Clearly the basic idea of this paper is applicable in other situations. A change in the number of states yields a class of algorithms for n-coloring planar maps or graphs in general. For example, taking the number of states to be two, we can use the algorithm to two-color a checker board. While two-coloring a checker board is very simple, this gives us a rudimentary way of measuring the efficiency of the algorithm. Experiments indicate that a 8×8 checkerboard is two-colored in about 32 generations on average. A 16×16 checkerboard is two-colored in about 82 generations on average. This seems efficient considering that an $n \times n$ checkerboard has 2^{n^2} distinct colorings by two colors, only two of which are two-colorings. The algorithm may also be used to four color maps. A map of the United States is four colored in about 114 generations on average.

While the algorithm is broadly applicable, it does not seem quite as good as more specifically designed algorithms. The recursive algorithm based on Kempe chains described in chapter 24 of [6] four-colors most maps a little faster, for example. Furthermore, all of the maps described here so far have adjacency graphs all of whose vertices have small degree. More complicated maps (like Martin Gardner's April fools map described in [6]) can stump the algorithm completely.

5 Implementation notes

All the images for this paper were generated with *Mathematica*. The tilings were generated using the **DigraphFractals** *Mathematica* package by the author as described in [4] These images were then converted to **PlanarMap** and **PlanarGraph** objects as defined in the **GraphColoring** *Mathematica* package by Stan Wagon [6]. Code to run the cellular automaton on the **PlanarGraph** objects was written by the author. Final three-colored images were rendered by the **ShowMap** function defined in the **GraphColoring** package.

All code, dynamic images, and more examples are available on the author's web page:
http://www.unca.edu/˜mcmcclur/mathematicaGraphics/PenroseColoring/

References

[1] Babilon, R., 3-colourability of Penrose kite-and-dart tilings. *Discrete Mathematics,* 2000, **235**, 137-143.

[2] Gardner, M., *Penrose Tiles to Trapdoor Ciphers*, W. H. Freeman, New York, 1989.

[3] Grünbaum, B. and Shepard, G. C., *Tilings and Patterns*, W. H. Freeman, New York, 1987.

[4] McClure, M., Digraph self-similar sets and aperiodic tilings. Submitted to *The Mathematical Intelligencer.*

[5] Sibley, T. and Wagon, S., Rhombic Penrose tilings can be 3-colored. *The American Mathematical Monthly*, 2000, **107**, 251-253.

[6] Wagon, S., *Mathematica in Action, 2 ed.*, Springer-Verlag, New York, 1999.

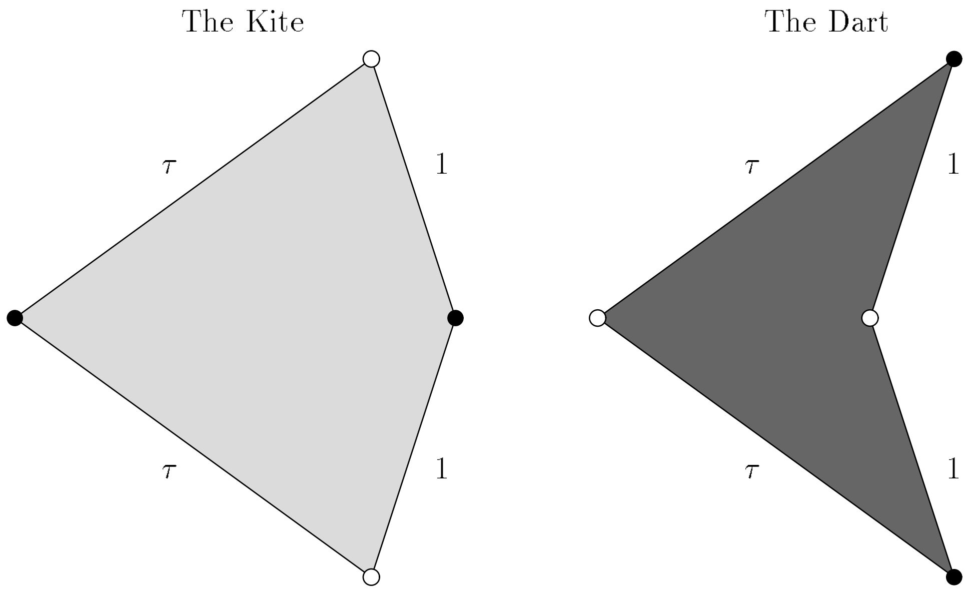

Figure 1: The kite and dart

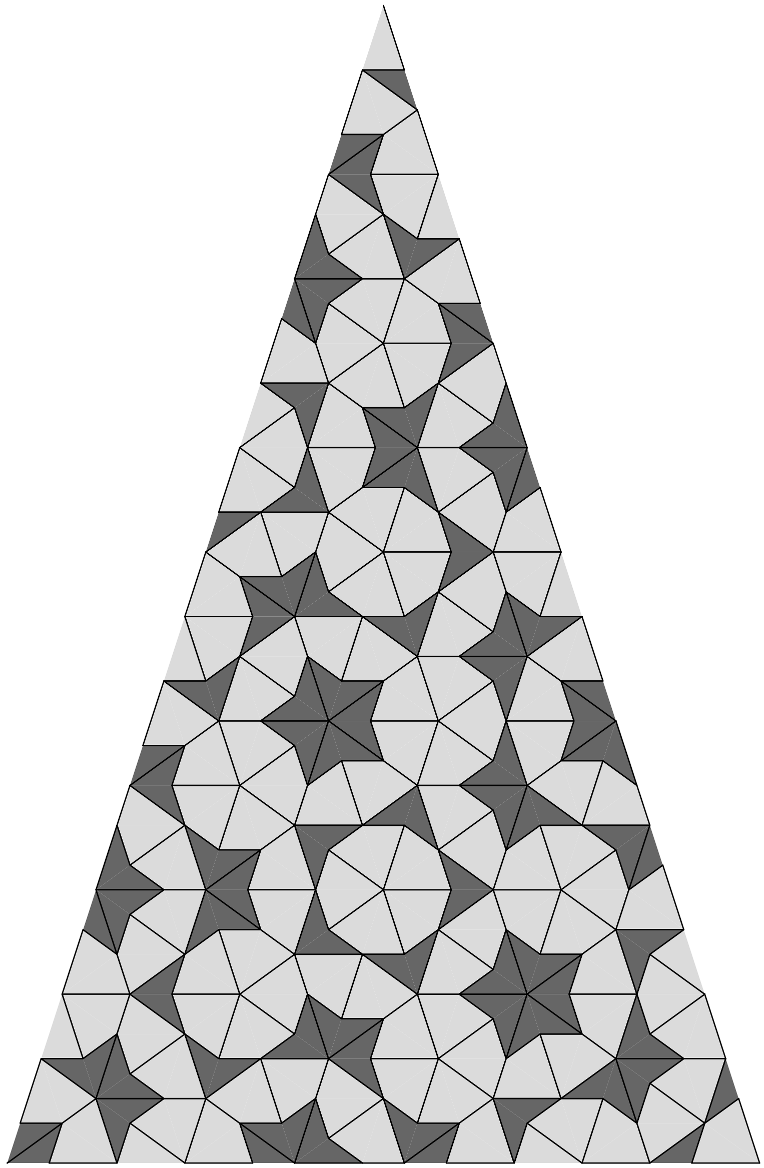

Figure 2: Part of a tiling by kites and darts

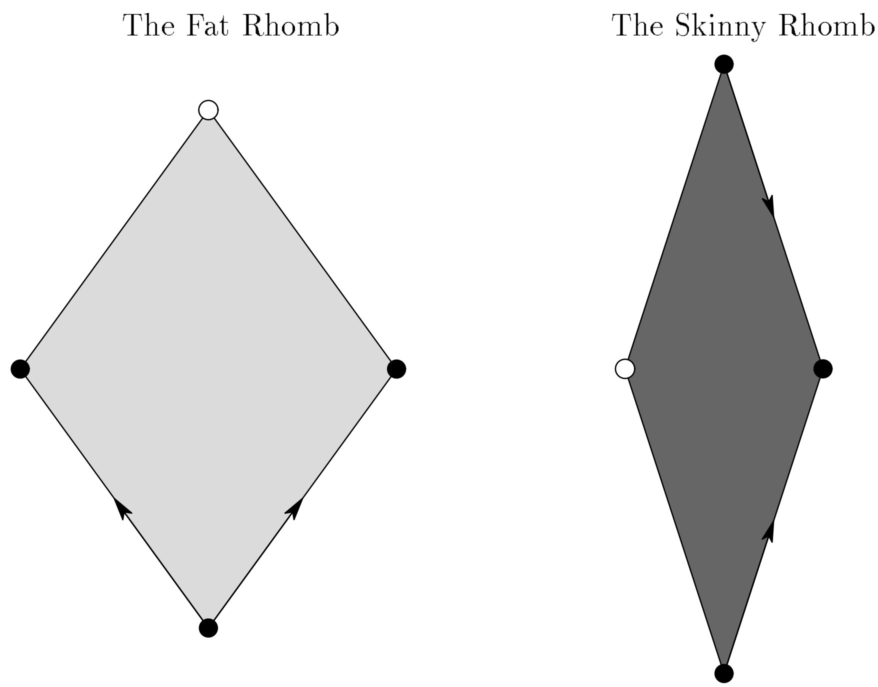

Figure 3: The fat and skinny rhombs

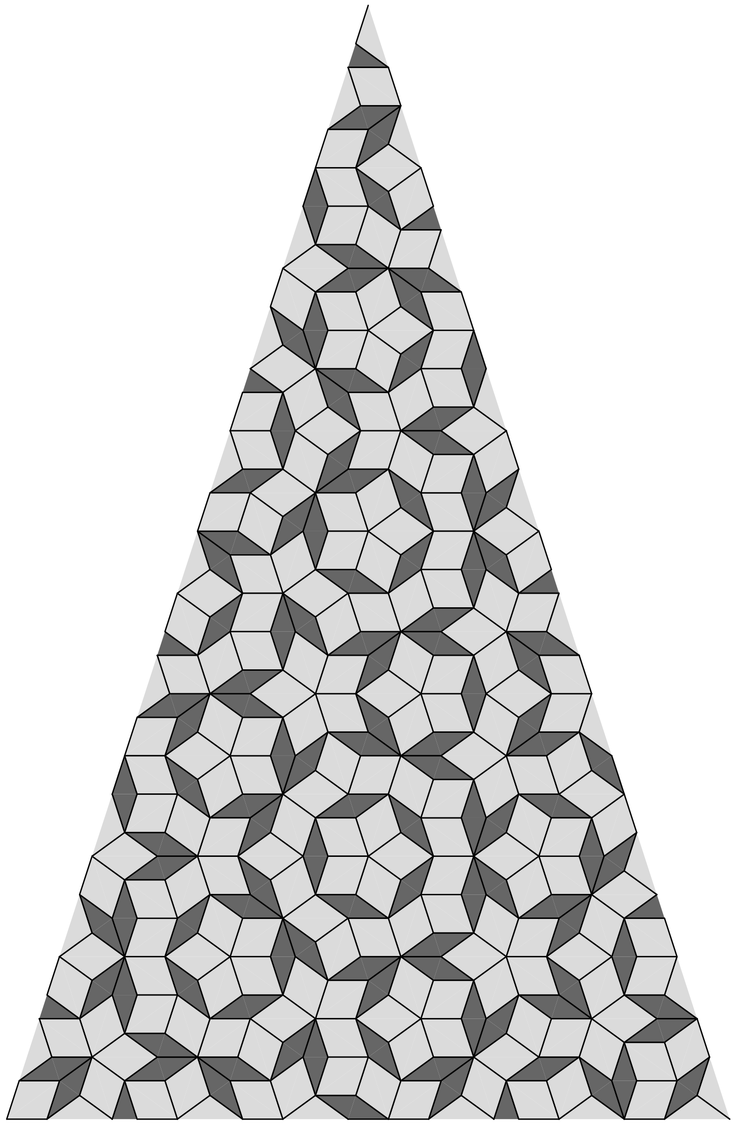

Figure 4: Part of a tiling by rhombs

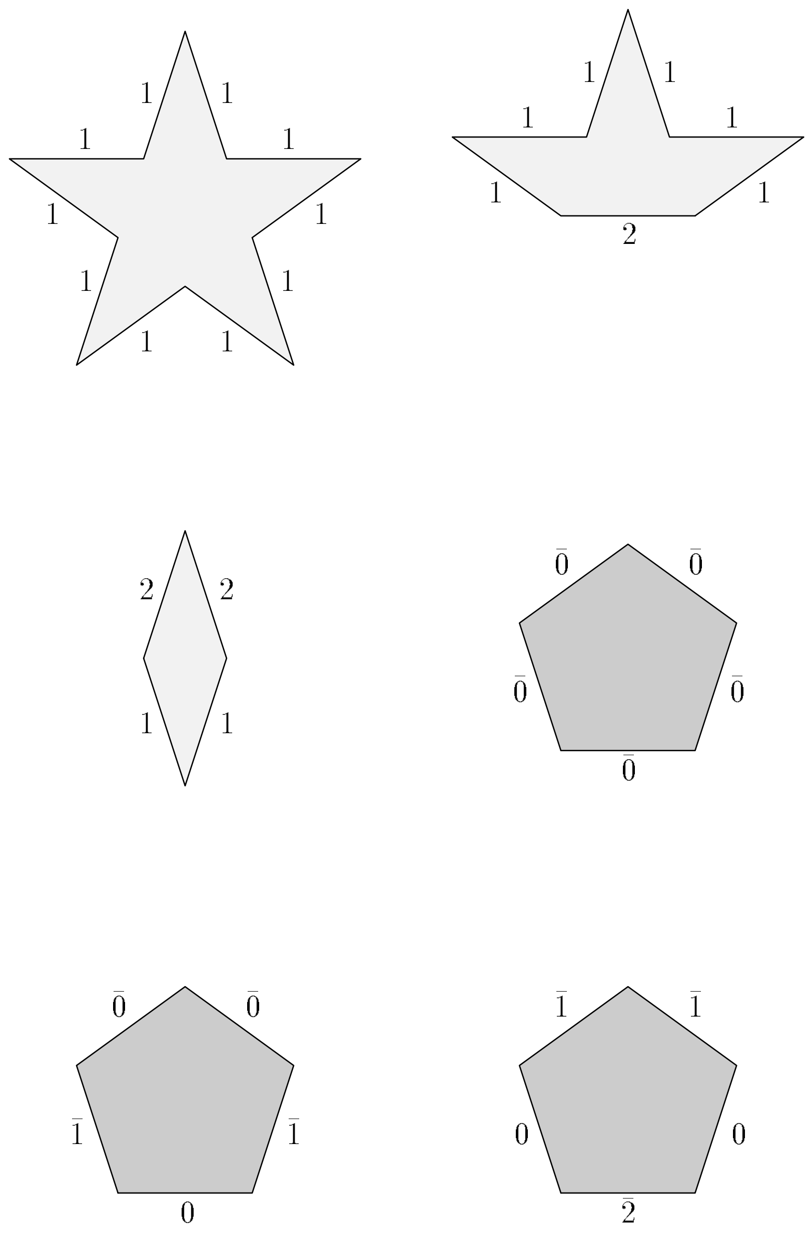

Figure 5: The pentacles

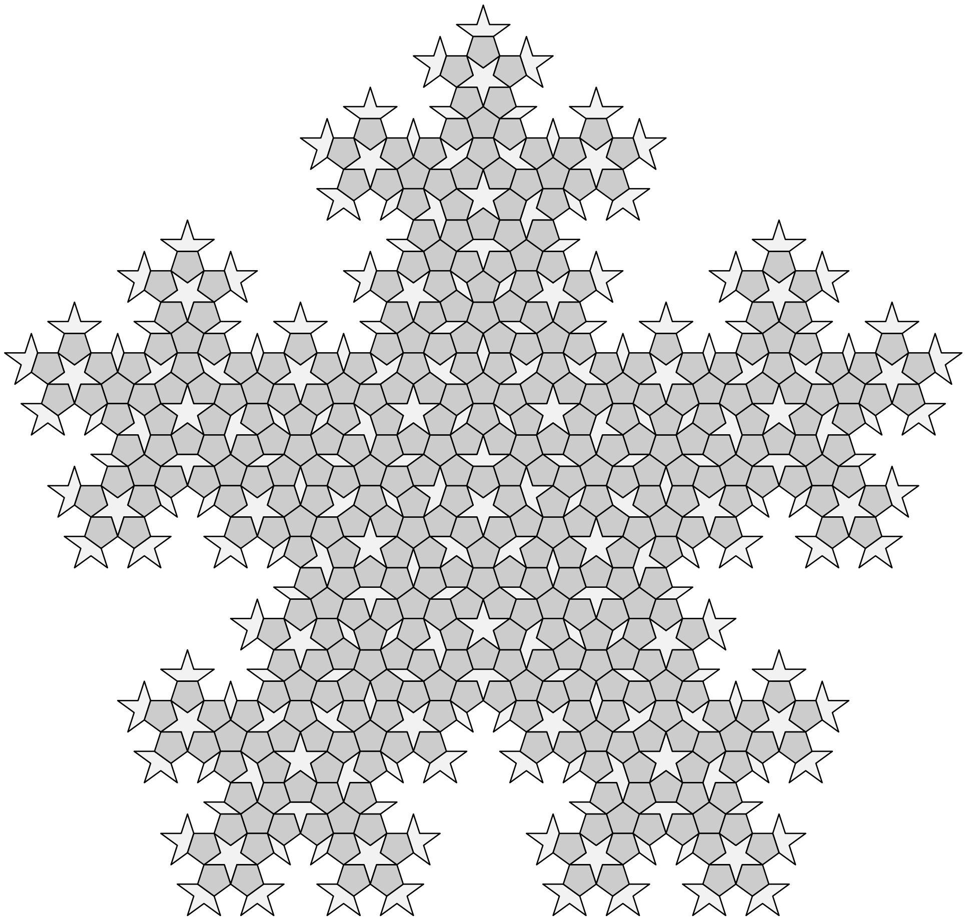

Figure 6: Part of a tiling by pentacles

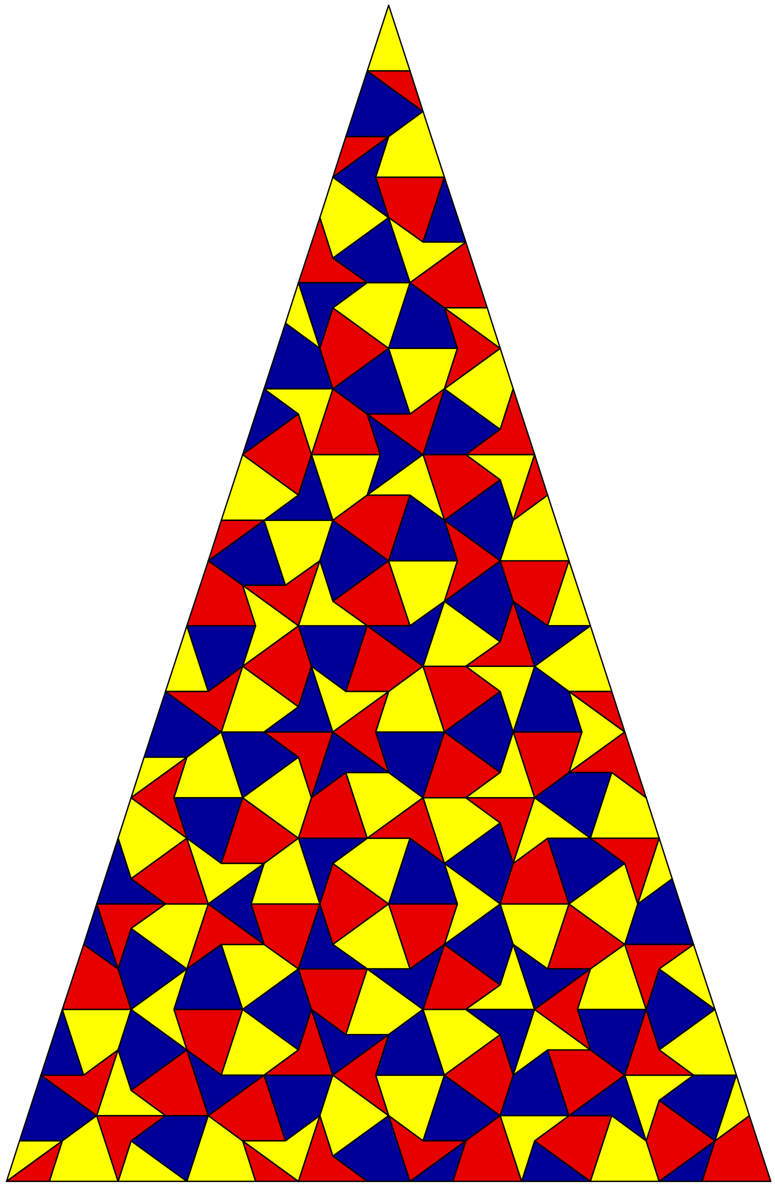

Figure 7: Part of a three-colored tiling by kites and darts

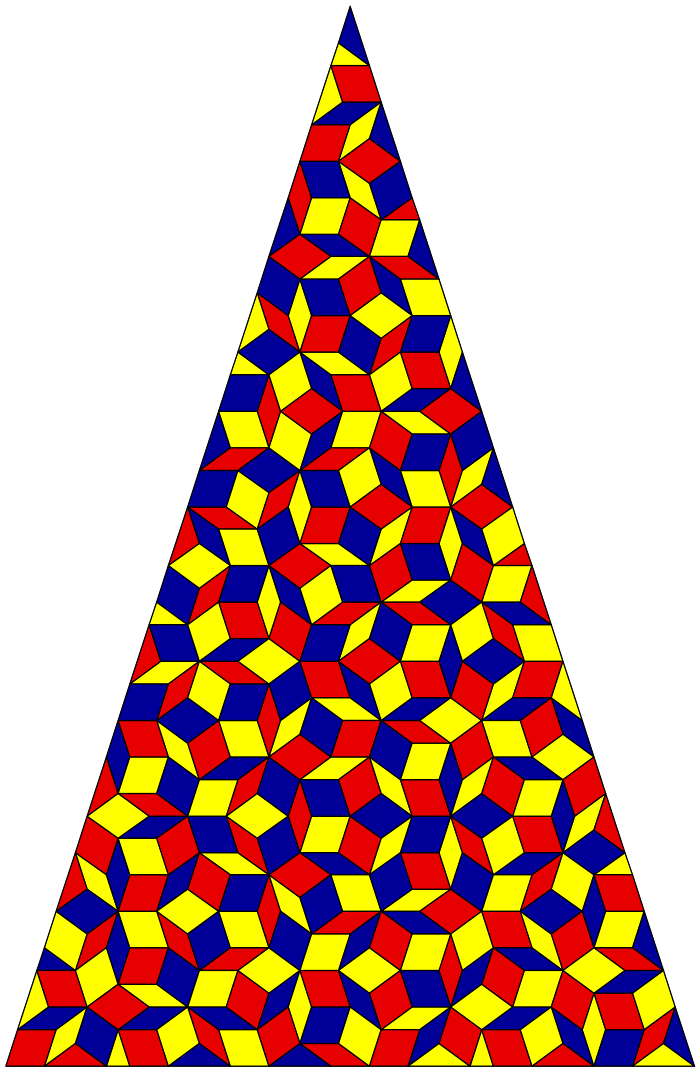

Figure 8: Part of a three-colored tiling by rhombs

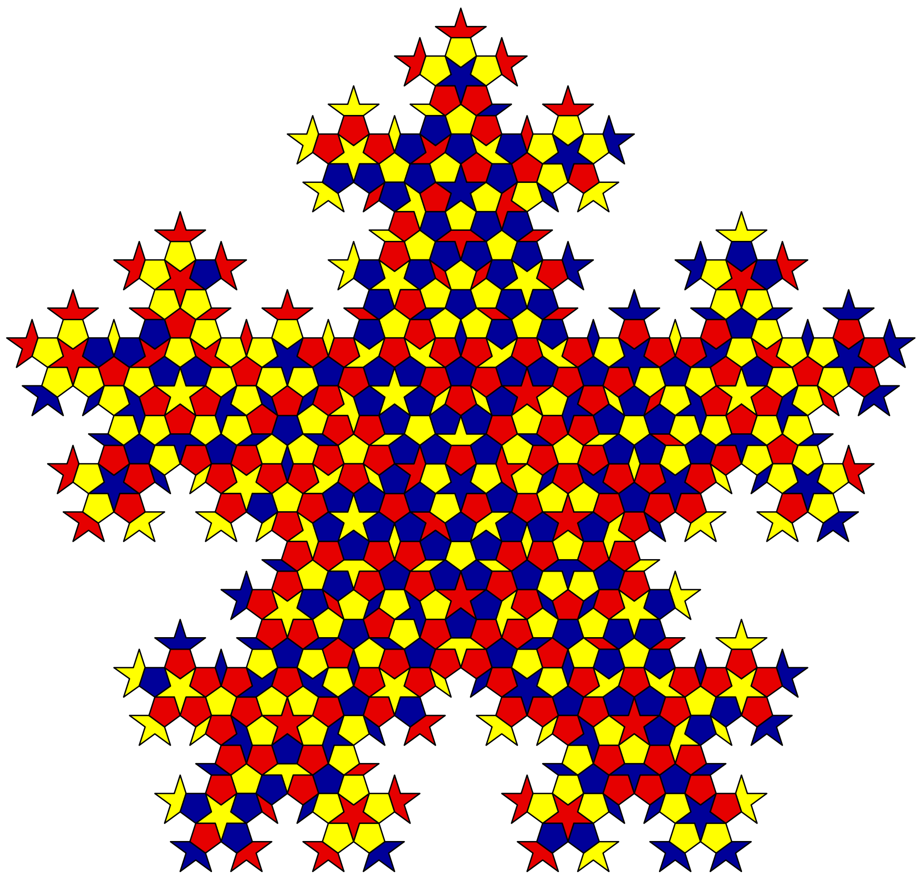

Figure 9: Part of a three-colored tiling by pentacles

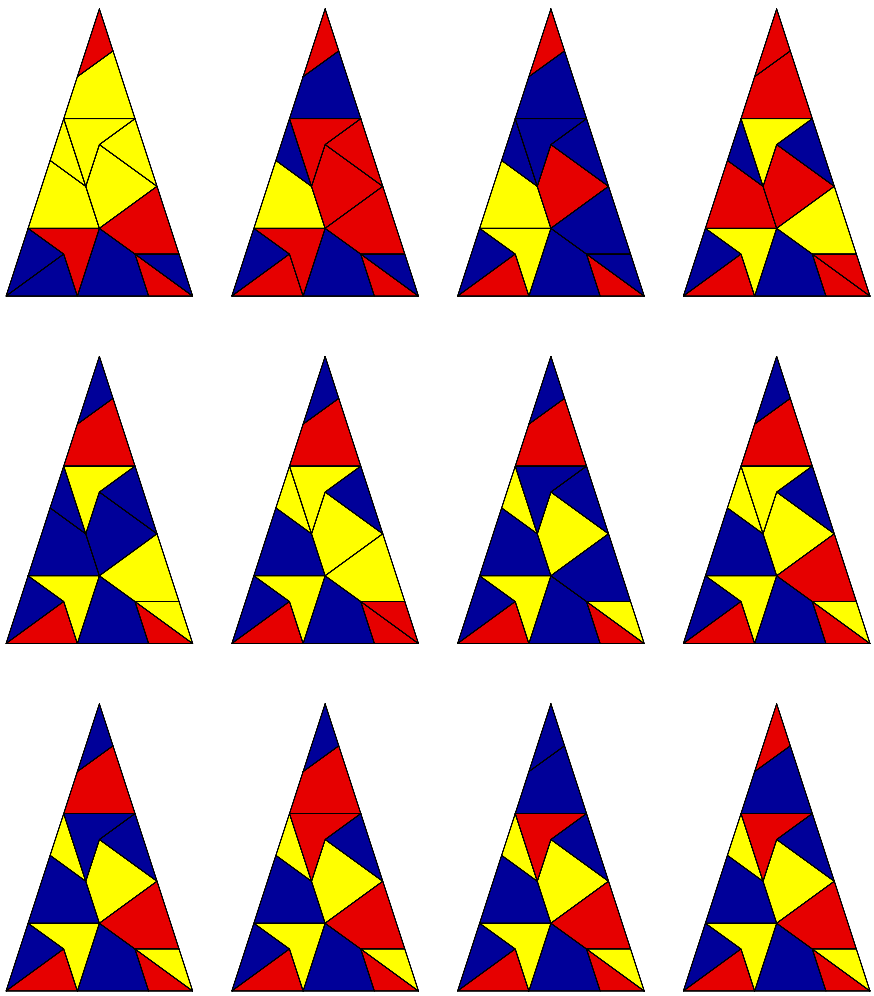

Figure 10: Evolution to a three-coloring