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# Directed-Graph Iterated Function Systems

by Mark McClure

Directed-graph iterated function systems are an important generalization of iterated function systems. In this article, we discuss their basic theory and implementation.

## Introduction

Consider the curve in the plane  $K$  shown in Figure 1.  $K$  is called the *Koch curve* and is an example of a self-similar set. Figure 1 shows how  $K$  is composed of 4 copies of itself, each scaled by a factor of  $\frac{1}{3}$ . More generally, a self-affine set is one composed of affine images of the whole set. The terminology *iterated function system*, or *IFS*, was introduced by Barnsley to describe a process for rendering images of such sets. There are many good references for IFSs including [Barnsley 1993], [Edgar 1990], and [Gutierrez, Iglesias, Rodriguez, and Rodriguez 1997] which describes an implementation in *Mathematica*.

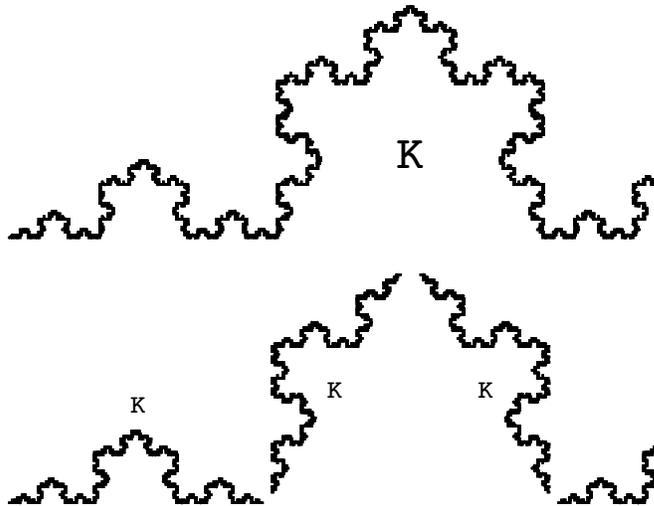


Figure 1

In this article, we consider a generalization of self-similar sets and its implementation in *Mathematica*. All of our functions are encapsulated in the package **DigraphFractals**. Consider the two curves  $A$  and  $B$

shown in figure 2.  $A$  is composed of 1 copy of itself, scaled by a factor  $\frac{1}{2}$ , and 2 copies of  $B$ , rotated and scaled by a factor  $\frac{1}{2}$ .  $B$  is composed of 1 copy of itself, scaled by a factor  $\frac{1}{2}$ , and 1 copy of  $A$ , reflected and scaled by a factor  $\frac{1}{2}$ . The sets  $A$  and  $B$  exhibit directed-graph self-similarity and may be described using a directed-graph iterated function system, or digraph IFS. Our formulation of a digraph IFS first appeared in [Mauldin and Williams, 1988] and is also described in [Edgar 1990] and [Falconer 1997].

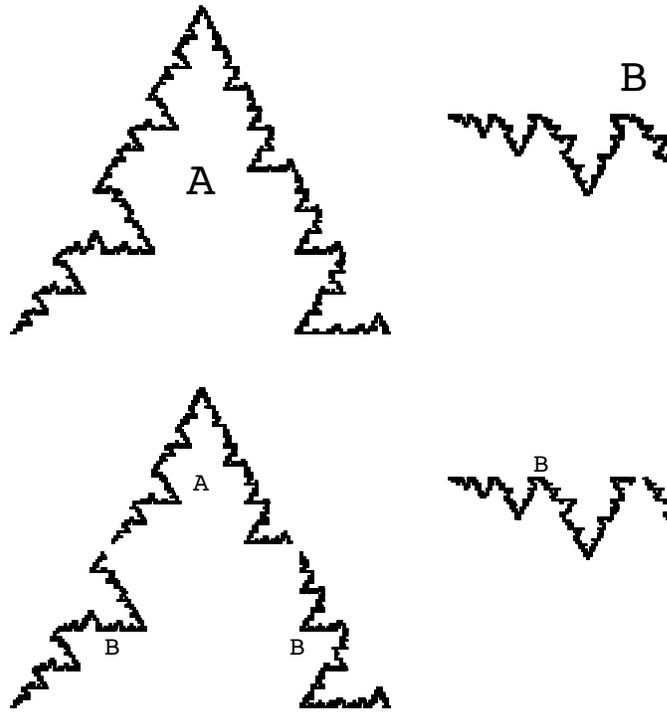


Figure 2

## Basic Theory

The first ingredient for a digraph IFS is a *directed multi-graph* which describes the combinatorics of how the pieces fit together. A directed multi-graph  $G$  consists of a finite set  $V$  of vertices and a finite set  $E$  of directed edges between vertices.  $G$  is called a multi-graph because we allow more than one edge between any two vertices. Given two vertices,  $u$  and  $v$ , we denote the set of all edges from  $u$  to  $v$  by  $E_{uv}$ . More generally, a *path* through  $G$  is a finite sequence of edges so that the terminal vertex of any edge is the initial vertex of the subsequent edge.  $G$  is called *strongly connected* if for every  $u$  and  $v$  in  $V$ , there is a path from  $u$  to  $v$ . We denote the set of all paths of length  $n$  with initial vertex  $u$  by  $E_u^n$ .

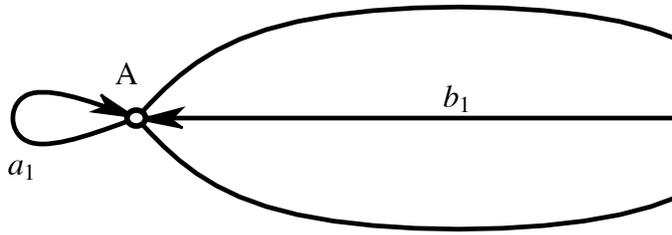


Figure 3

Figure 3 shows the directed multi-graph for the curves  $A$  and  $B$ . There are two edges from node  $A$  to node  $B$  and one edge from node  $A$  to itself since  $A$  consists of two copies of  $B$  together with one copy of itself. Similarly, there is one edge from node  $B$  to node  $A$  and one edge from  $B$  to itself since  $B$  consists of one copy of  $A$  together with one copy of itself.

Note that the edges in figure 3 are labeled. In general, we obtain a *digraph IFS* from a directed multi-graph by associating an affine function  $f_e$  with each edge  $e$  of the digraph. There are two main types of functions we will consider. A function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is called a *similarity transformation*, or simply a similarity, if there is an  $r > 0$ , called the similarity ratio of  $f$ , such that  $|f(\mathbf{x}) - f(\mathbf{y})| = r|\mathbf{x} - \mathbf{y}|$  for all  $\mathbf{x}$  in  $\mathbb{R}^n$ . More generally, an *affine transformation* is a linear transformation plus a translation. The labels in figure 3 correspond to the following affine functions, which are in fact similarities.

$$\begin{aligned} \mathbf{a}_1(\mathbf{x}) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix} & \mathbf{a}_2(\mathbf{x}) &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} \\ \mathbf{a}_3(\mathbf{x}) &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{3}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix} \\ \mathbf{b}_1(\mathbf{x}) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} & \mathbf{b}_2(\mathbf{x}) &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbf{x} \end{aligned}$$

An affine transformation need not scale a set by the same amount in all directions. However, we may characterize its largest scaling factor by the *spectral radius* of the linear part, which is the maximum of the absolute values of all its complex eigenvalues. A digraph IFS is called *contractive* if the product of the scaling factors in any closed loop is less than 1. Theorem 4.3.5 of [Edgar 1990] states that for any contracting digraph IFS, there is a unique set of closed, bounded sets  $K_v$ , one for every  $v$  in  $V$ , such that for every  $u$  in  $V$ ,

$$K_u = \bigcup_{v \in V, e \in E_{uv}} f_e(K_v). \quad (1)$$

Such a collection of sets is called the *invariant list* of the digraph IFS and we will refer to its members as *digraph fractals*. Note that an IFS is the special case of a digraph IFS with one node and theorem 4.3.5 of [Edgar 1990] is a generalization of the statement that a unique invariant set exists for a contractive IFS.

## Implementation of a Deterministic Algorithm

In this section, we implement a deterministic algorithm to generate the sets  $A$  and  $B$ . The first step is to determine an appropriate data structure to describe a digraph IFS. For this purpose, we will use an adjacency matrix of lists of affine functions. The element in row  $u$  and column  $v$  will be a list of affine functions which map  $E_v$  into  $E_u$ . Each affine function will be described by a pair  $\{A, b\}$ , where  $A$  is a two dimensional matrix

indicating the linear part and  $b$  is a two dimensional vector indicating the shift. Here is the code describing the digraph IFS for the curves  $A$  and  $B$ .

```

rotate[θ_] := {{Cos[θ], -Sin[θ]},
              {Sin[θ], Cos[θ]}};
a1 = {{{1/2, 0}, {0, 1/2}}, {1/4, √3/4}};
a2 = {1/2 rotate[π/3], {0, 0}};
a3 = {1/2 rotate[-π/3], {3/4, √3/4}};
b1 = {{{1/2, 0}, {0, -1/2}}, {1/2, 0}};
b2 = {{{1/2, 0}, {0, 1/2}}, {0, 0}};
curvesDigraph = {{{a1}, {a2, a3}}, {{b1}, {b2}}};

```

We also need an initial approximation to each digraph fractal for the digraph IFS to act upon. In general, these could be any **Graphics** primitives, but some objects lead to more natural approximations than others. Note that each initiator is a list of **Graphics** primitives. This way each initiator may be a fairly complicated graphic. Here is our choice of initiators.

```

lineSegment = Line[{{0, 0}, {1, 0}}];
initiators = {{lineSegment}, {lineSegment}};

```

Each affine list needs to be transformed into a function which acts on the initiators. The version of **toFuncs** used by the package takes **Graphics** primitives other than **Line** into account.

```

toFuncs[{{A_, b_}} := Module[{fOut},
  fOut[{{x_, y_}}] := N[A.{x, y} + b];
  fOut[Line[x_]] := Line[fOut /@ x];
  fOut
];
digraphFuncs = Map[toFuncs, curvesDigraph, {3}];

```

We now need to describe the action of the digraph IFS on the initiators to get to the next stage in the approximation. Suppose that  $K_u^0$  is the initiator for  $K_u$ . Then by equation (1), we need to replace  $K_u^0$  with the union over all  $v \in V$  and  $e \in E_{uv}$  of  $f_e(K_v^0)$ . Now the element in row  $u$  and column  $v$  of our digraph IFS contains a list of all  $f_e$  for  $e \in E_{uv}$ . Thus we need to apply all of the functions in this list to  $K_v^0$  (using **Outer**) and take the union over all  $v \in V$ . Note that this is practically matrix multiplication, or more correctly an inner product. There is a complication. The function **Inner** gives a generalized inner product which goes deeply into nested structures. We will need to change the heads at the appropriate level to prevent this. Taking all of this into account, we obtain the following code for the first step.

```

digraphFuncs = Apply[placeholder, digraphFuncs, {2}];
initiators = Apply[placeholder, initiators, {1}];
step =
  Inner[Outer[#1[#2] &, #1, #2] &, digraphFuncs, initiators, List];

```

This process is now repeated with **step** as the new input to get to the next stage.

```

step = Inner[Outer[#1[#2] &, #1, #2] &, digraphFuncs, step, List];

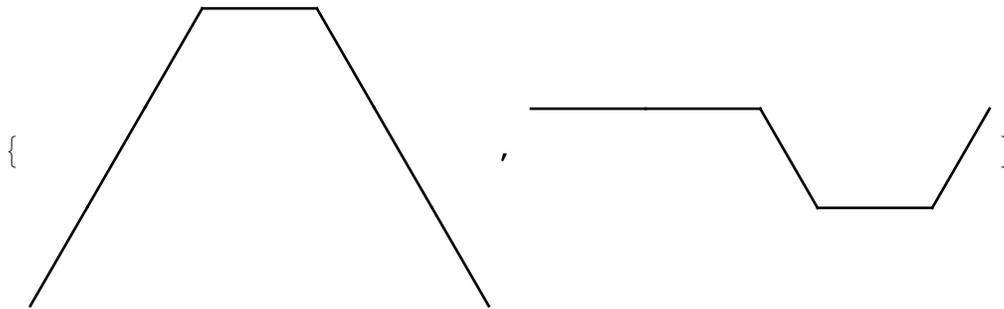
```

Note that **step** now holds a list of lists of graphics primitives held in **placeholders**. To see the approximation we simply get rid of the **placeholders** and map **Show** onto the list.

```

graphicList = step //. placeholder[x___] -> {x};
Show[Graphics[#], AspectRatio -> Automatic] & /@ graphicList

```

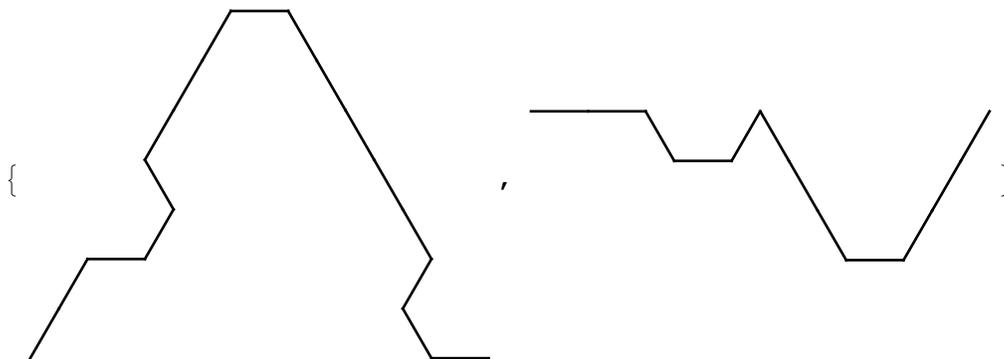


Here is the next step.

```

step = Inner[Outer[#1[#2] &, #1, #2] &, digraphFuncs, step, List];
graphicList = step //. placeholder[x___] -> {x};
Show[Graphics[#], AspectRatio -> Automatic] & /@ graphicList

```



To see a higher order approximation, we simply **Nest** this procedure.

```

depth = 10;
graphicList =
  Nest[Inner[Outer[#1[#2] &, #1, #2] &, digraphFuncs, #, List] &,
    initiators, depth] //. placeholder[x_>>] -> {x};
Show[Graphics[#], AspectRatio -> Automatic] & /@ graphicList

```



## Examples

The algorithm described in the previous section is encapsulated in our package function **ShowDigraphFractals**, which generates images in the plane. In this section, we illustrate the use of this function with several more examples. First, we load the **DigraphFractals** package.

```
Needs["FractalGeometry`DigraphFractals`"];
```

Here is the package description of the function **ShowDigraphFractals**.

```
? ShowDigraphFractals
```

ShowDigraphFractal[digraph\_,depth\_] generates the approximation to the digraph self similar sets defined by digraph to order depth using a deterministic algorithm. The digraph is represented by a matrix of lists of affine functions written in the form  $\{\{a_1, a_2\}, \{b_1, b_2\}, \{x_0, y_0\}\}$ .

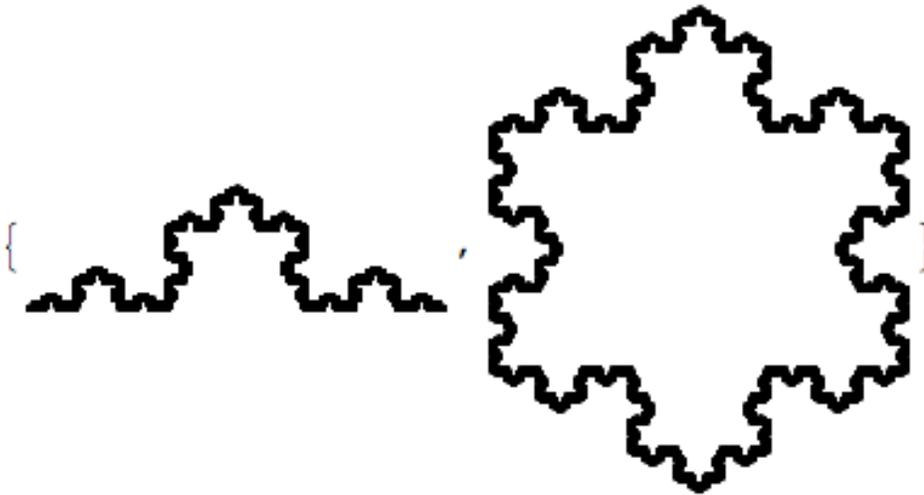
### ■ The Koch Snowflake

The Koch curve is an example of a self-similar set and may be realized as a special case of a digraph fractal. More than this, three copies of the Koch curve form the boundary of a set known as the Koch snowflake. Thus both sets together may be realized as the invariant list of the following digraph IFS.

```

c1 = {{{1/3, 0}, {0, 1/3}}, {0, 0}};
c2 = {rotate[ $\pi/3$ ]/3, {1/3, 0}};
c3 = {rotate[- $\pi/3$ ]/3, {1/2,  $\sqrt{3}/6$ }};
c4 = {{{1/3, 0}, {0, 1/3}}, {2/3, 0}};
f1 = {{{1, 0}, {0, -1}}, {0, 0}};
f2 = {rotate[ $\pi/3$ ], {0, 0}};
f3 = {rotate[- $\pi/3$ ], {1/2,  $\sqrt{3}/2$ }};
Koch = {{{c1, c2, c3, c4}, {}}, {{f1, f2, f3}, {}}};
ShowDigraphFractals[Koch, 7,
  Prolog -> {AbsolutePointSize[.4]}]

```



A few observations are in order. First, the digraph above is not strongly connected. There is no path from the curve  $C$  to the flake  $F$ . Nonetheless,  $F$  is a simple union of copies of  $C$  and **ShowDigraphFractals** works as expected. Second, note that no initiators were specified. The option **Initiators** may be used to define the initiators. The default, **Initiators**→**Automatic**, uses the single point  $\{0, 0\}$  as the initiator for each set. Finally, note that **ShowDigraphFractals** accepts the usual **Graphics** options

## ■ Highway's Dragon

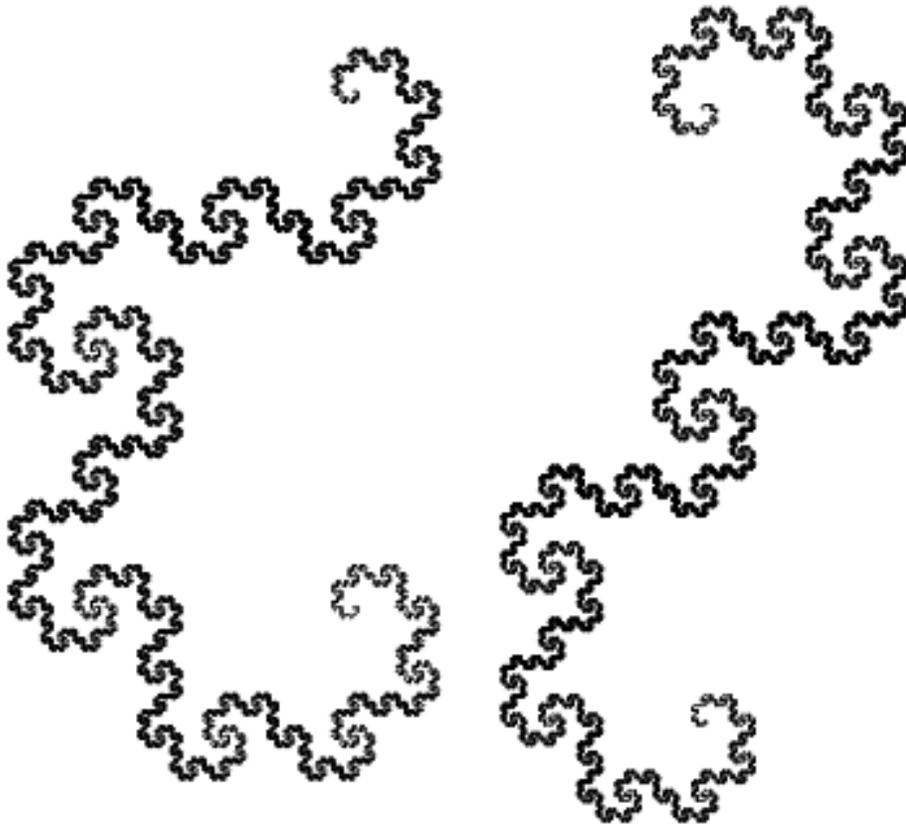
Highway's dragon is a self-similar set in the plane. Its boundary consists of the two digraph fractals defined here. This time we use two vertical line segments as the initiators.

```

f1 = {{{1, 0}, {0, 1}}.rotate[ $\pi/4$ ]/ $\sqrt{2}$ , {0, 0}};
f2 = {-rotate[- $\pi/4$ ]/ $\sqrt{2}$ , {0, 1}};
g1 = {{{1/2, 0}, {0, 1/2}}, {0, 0}};
g2 = {{{-1/2, 0}, {0, -1/2}}, {0, 1}};
highwayDragon = {{{f1}, {f2}}, {{g1, g2}, {}}};
lineSegment = Line[{{0, 0}, {0, 1}}];
highwayBoundary = ShowDigraphFractals[highwayDragon, 14,
  Initiators -> {{lineSegment}, {lineSegment}},
  DisplayFunction -> Identity];
Show[GraphicsArray[highwayBoundary]]

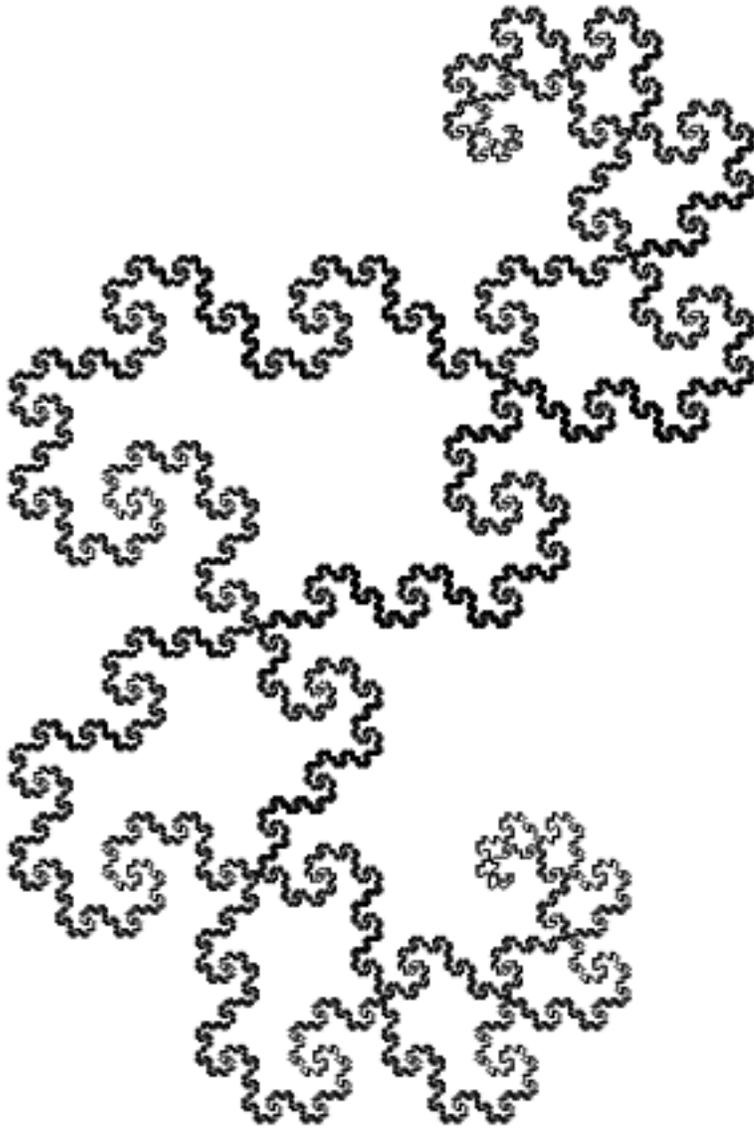
```

— *GraphicsArray:obs: GraphicsArray is obsolete Switching to GraphicsGrid >>*



Heighway's dragon boundary may be rendered by showing these two curves together.

```
Show[heighwayBoundary,  
      DisplayFunction -> $DisplayFunction]
```



Alternatively, we could set up a non-strongly connected digraph IFS with this third set corresponding to the third node, just as we did for the Koch snowflake.

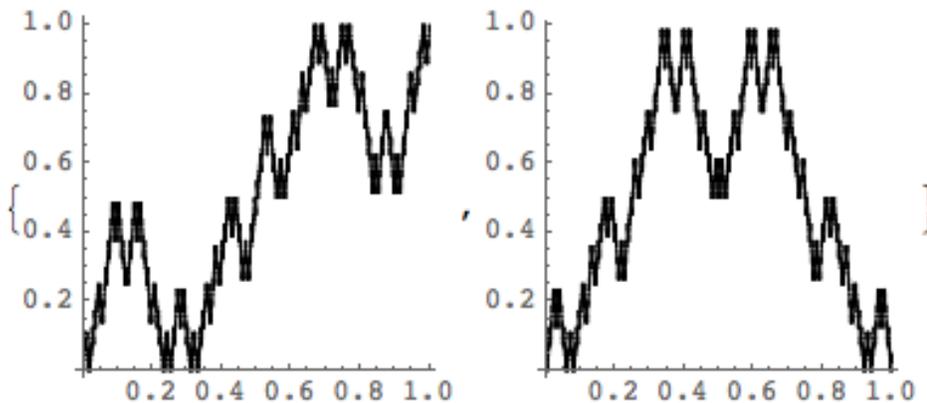
## ■ Hilbert's Curve

Hilbert's space filling curve is a continuous function that maps the unit interval onto the unit square. It turns out that the coordinate functions of this curve may be described using the following digraph IFS.

```

A = {{1/4, 0}, {0, 1/2}}; B = {{1/4, 0}, {0, -1/2}};
f1 = {A, {0, 0}}; f2 = {A, {1/4, 0}};
f3 = {A, {1/2, 1/2}}; f4 = {B, {3/4, 1}};
g1 = {A, {0, 0}}; g2 = {A, {1/4, 1/2}};
g3 = {A, {1/2, 1/2}}; g4 = {B, {3/4, 1/2}};
hilbertCurves =
  {{{f2, f3}, {f1, f4}}, {{g1, g4}, {g2, g3}}};
initiators =
  {{Line[{{0, 0}, {1, 1}}]}, {Line[{{0, 0}, {1, 0}}]}};
ShowDigraphFractals[hilbertCurves, 6,
  Initiators -> initiators, Axes -> True]

```



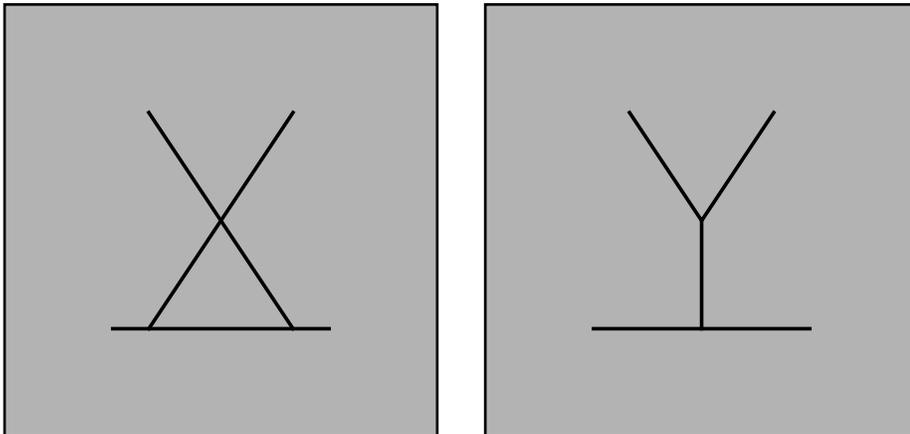
Note that the two initiators above were different. In general we may use much more complicated initiators. This may be used to illustrate the effect of the digraph IFS. For example, consider the following initiators.

```

initiators =
  {{ {GrayLevel[.7],
    Polygon[{{0, 0}, {1, 0}, {1, 1}, {0, 1}}]},
    Line[{{0, 0}, {1, 0}, {1, 1}, {0, 1}, {0, 0}}],
    {Thickness[.008],
    Line[{{1/3, 1/4}, {2/3, 3/4}}]},
    Line[{{1/3, 3/4}, {2/3, 1/4}}]},
    Line[{{1/4, 1/4}, {3/4, 1/4}}]}},
  {{ {GrayLevel[.7],
    Polygon[{{0, 0}, {1, 0}, {1, 1}, {0, 1}}]},
    {Thickness[.008],
    Line[{{1/3, 3/4}, {1/2, 1/2}, {2/3, 3/4}}]},
    Line[{{1/2, 1/2}, {1/2, 1/4}}]},
    Line[{{1/4, 1/4}, {3/4, 1/4}}]},
    Line[{{0, 0}, {1, 0}, {1, 1}, {0, 1}, {0, 0}}]}];
Show[GraphicsArray[
  Graphics[#, AspectRatio -> Automatic] & /@ initiators]]

```

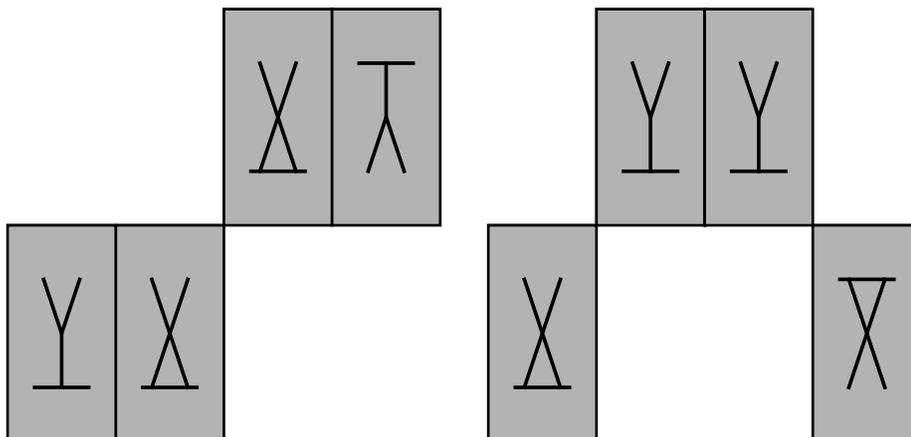
— *GraphicsArray:obs: GraphicsArray is obsolete Switching to GraphicsGrid >>*



The action of the digraph IFS may now be easily discerned from the following image

```
Show[GraphicsArray[
  ShowDigraphFractals[hilbertCurves, 1,
    Initiators -> initiators,
    DisplayFunction -> Identity]]]
```

— *GraphicsArray:obs: GraphicsArray is obsolete Switching to GraphicsGrid >>*



A proof that these curves are indeed the coordinate functions of Hilbert's curve may be found in [McClure 1999]. For those familiar with Hilbert's curve, here is a convincing graphical argument. We first generate approximations to our curves using carefully chosen initiators.

```
twoPoints = {{Point[{1/2, 1/2}]},
  {Point[{1/2, 1/2}]}];
approximations =
  ShowDigraphFractals[hilbertCurves, 4,
    Initiators -> twoPoints,
    DisplayFunction -> Identity];
```

We can get the actual points as follows

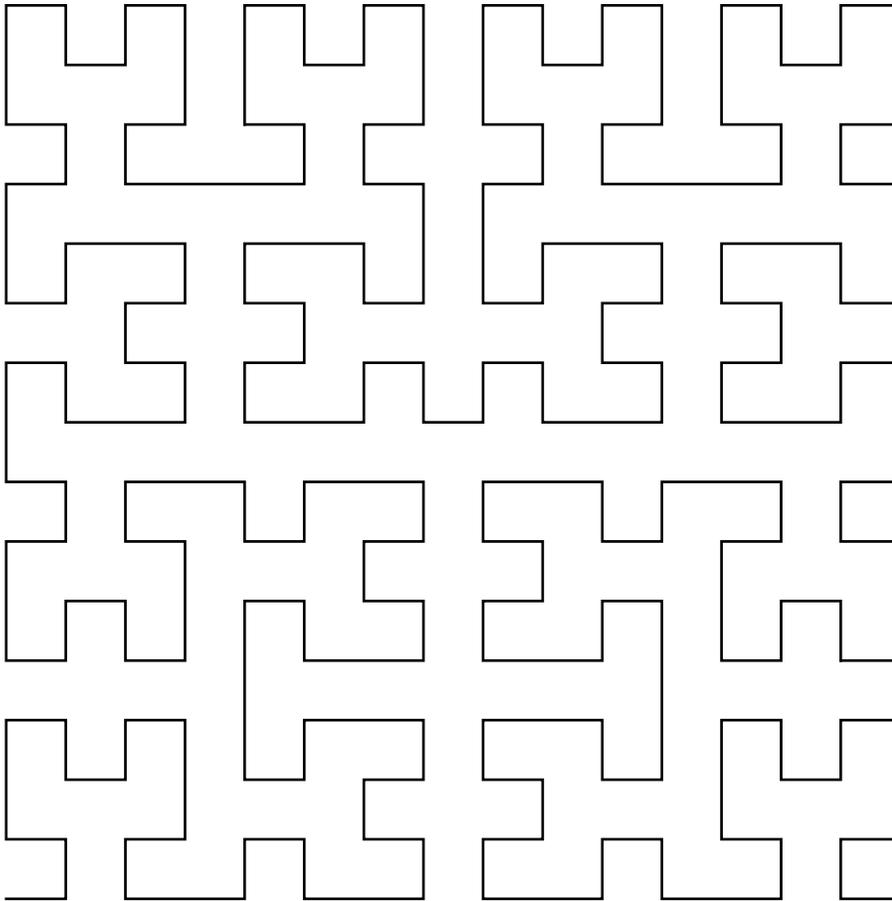
```
(* Modified for multi-point format *)
graphPoints =
  Join@@Cases[#, Point[xs_] :-> xs, Infinity] & /@ approximations;
```

We then sort these points, collect a list of  $x$  versus  $y$ , and draw a polygonal line through the resulting list of points. The result is a familiar approximation to Hilbert's curve.

```

sortedGraphPoints =
  Sort[#, First[#1] < First[#2] &] & /@
  graphPoints;
curvePoints =
  Transpose[sortedGraphPoints /.
    {x_Real, y_Real} -> y];
Show[Graphics[Line[curvePoints]],
  AspectRatio -> Automatic]

```



## A Stochastic Algorithm

We now describe a stochastic algorithm similar to the random iteration algorithm for an IFS. We begin by assigning a positive probability  $p_e$  to every  $e \in E$  such that for every  $v \in V$ ,

$$\sum_{e \in E_{uv}, u \in V} p_e = 1.$$

We then choose an arbitrary point  $x \in \mathbb{R}^2$ , an arbitrary vertex  $v \in V$ , and some  $e \in E_{uv}$  according to our probability list  $\{p_e : e \in E_{uv}, u \in V\}$  and apply  $f_e$  to  $x$ . If  $e \in E_{uv}$ , this gives us a point  $f_e(x)$  in our approximation to  $K_u$ , which then becomes our new input and the process continues. We are essentially performing a random walk along the digraph and picking up points as we go along. The edges are traversed in reverse order since

$f_e : K_v \rightarrow K_u$  when  $e \in E_{uv}$ .

We implement this algorithm to generate the digraph fractal curves resulting from a mild variation of our first example. The main point of the variation is to illustrate the effect of differences in the similarity ratios. We first define the digraph and transform it into a digraph containing pure functions. We also define a probability matrix **pMatrix** satisfying the conditions described above. Note that we need the sum of all of the probabilities of all of the edges coming in to any particular node to be 1. For our matrix representation of the digraph, this means that the sum of all of the probabilities in any column should be 1. For this reason, it will be convenient to transpose the **pMatrix** since it is easier to access rows than to access columns. In this first example, we choose these probabilities to be somewhat arbitrary.

```

a1 = {{{3/4, 0}, {0, 3/4}}, {1/8, sqrt(3)/8}};
a2 = {1/4 rotate[pi/3], {0, 0}};
a3 = {1/4 rotate[-pi/3], {7/8, sqrt(3)/8}};
b1 = {{{3/4, 0}, {0, -3/4}}, {1/4, 0}};
b2 = {{{1/4, 0}, {0, 1/4}}, {0, 0}};
newCurvesDigraph = {{{a1}, {a2, a3}}, {{b1}, {b2}}};
toFuncs[{A_, b_}] := N[A.# + b] &;
digraphFuncs = Map[toFuncs, newCurvesDigraph, {3}];
pMatrix = N[{{1/2}, {1/4, 1/4}}, {{1/2}, {1/2}}];

```

We now need a method to use **pMatrix** to choose edges according to the scheme described above. This is based in part on the following technique for choosing a random integer according to a list of probabilities **pList**.

```

pList = {.1, .2, .3, .4};
foldedPList = FoldList[Plus, 0, pList];
chooser = Random[];
Length[Select[foldedPList, (# < chooser) &]]

```

2

Suppose we are at some vertex  $v$  at a point along our random walk. We will move to the next vertex by first choosing which vertex to move to and then deciding which edge to travel along. The matrix **pMatrix1** will help us make the first of these decisions. We now transpose **pMatrix**, add all of the elements in the list in any row and column, and then use **FoldList** as above.

```

pMatrix1 = Apply[Plus, Transpose[pMatrix], {2}];
pMatrix1 = FoldList[Plus, 0, #] & /@ pMatrix1;

```

**pMatrix2** will help us make the second of our decisions. To get **pMatrix2**, we simply normalize the probabilities in the list in any row and column.

```

pMatrixNormalizer[subPList_List] :=
  FoldList[Plus, 0, subPList] / Plus @@ subPList;
pMatrix2 = Transpose[Map[pMatrixNormalizer, pMatrix, {2}]];

```

We now write a function **choose** which will accept a row from **pMatrix1** and the corresponding row from **pMatrix2** and decide which edge to take. **choose** will return an ordered pair of integers  $\{i, j\}$  where  $i$  indicates which vertex to travel to and  $j$  indicates which edge to take.

```

choose[v1_, v2_] := Module[
  {i, j, chooser},
  chooser = Random[];
  i = Length[Select[v1, (# < chooser) &]];
  chooser = Random[];
  j = Length[Select[v2[[i]], (# < chooser) &]];
  {i, j}];

```

We now set up an initial configuration and a place to store the points.

```

v = 1; v1 = pMatrix1[[v]]; v2 = pMatrix2[[v]];
currentPoint = {0, 0}; numPoints = 3000;
points = Table{}, {Length[newCurvesDigraph]}];

```

The following **Do** loop implements the algorithm using the tools we have described.

```

Do{
  {i, j} = choose[v1, v2];
  currentPoint = digraphFuncs[[i, v, j]][currentPoint];
  points[[i]] = {points[[i]], currentPoint};
  v = i; v1 = pMatrix1[[v]]; v2 = pMatrix2[[v]];
  }, {numPoints};

```

Finally, we pass these points to **ListPlot**.

```

points = Partition[Flatten[#, 2] & /@ points;
ListPlot[#,
  PlotStyle → {AbsolutePointSize[0.5]},
  AspectRatio → Automatic,
  Axes → False] & /@ points;

```

## ■ Determining the probabilities

Our technique seems to have worked, but clearly the images may be improved. This is because the points are not uniformly distributed throughout the set. In general, the distribution of the points in the approximation will depend upon the choice of probabilities in **pMatrix**. The question arises as to how to choose the probabilities to generate a uniform distribution, which renders the image more efficiently.

In the IFS theory, there is a notion of similarity dimension which coincides with the fractal (Hausdorff) dimension given certain additional assumptions. A self-similar set is generated by a digraph IFS with one node where to each  $e \in E$  corresponds a similarity  $f_e$ . If  $r_e$  is the similarity ratio of  $f_e$ , then the similarity dimension of the self-similar set is the unique  $s > 0$  such that

$$\sum_{e \in E} r_e^s = 1.$$

Perhaps not surprisingly, a choice of  $p_e = r_e^s$  is the correct way to choose the probabilities to generate this self-similar set efficiently. It's not unreasonable to suppose that the correct choice of a **pMatrix** for a digraph IFS will be connected the appropriate notion of dimension.

Suppose we have a digraph IFS such that to each  $e \in E$  corresponds a similarity  $f_e$  with similarity ratio  $r_e$ . For  $s > 0$ , define a numerical matrix  $A(s)$ , with rows and columns indexed by  $V$ , such that the entry in row  $u$  and column  $v$  is

$$A_{uv}(s) = \sum_{e \in E_{uv}} r_e^s.$$

The similarity dimension of the digraph IFS is defined to be the unique value of  $s$  such that the matrix  $A(s)$  has spectral radius 1. The main result in [Mauldin and Williams 1988] states that this dimension is equal to the Hausdorff dimension of each of the generated digraph fractals under certain additional assumptions. This theorem is also proved as theorems 6.4.8 and 6.6.6 in [Edgar 1990] and as corollary 3.5 in [Falconer 1997]. An understanding of this proof would help the interested reader understand why our technique for choosing the probabilities works. The important property of the similarity dimension is that  $s$ -dimensional Perron numbers for the digraph IFS exist only for this particular value of  $s$ . These are numbers  $q_v$ , indexed by  $V$ , such that

$$q_u^s = \sum_{e \in E_{uv}, v \in V} r_e^s q_v^s. \quad (2)$$

The numbers  $q_u^s$  simply form an eigenvector for the matrix  $A(s)$  corresponding to the eigenvalue 1. Note that if we take the transpose of the matrix  $A(s)$ , find the Perron numbers for this matrix, and then normalize the resulting equations by dividing through by  $q_u^s$ , we obtain what appears to be a legitimate **pMatrix1**. Taking the transpose simply makes  $A(s)$  fit into our algorithm and should pose no difficulty since the transpose preserves the spectral radius. This is exactly the strategy we will adopt to compute the probabilities

We now implement this plan to compute the matrices **pMatrix1** and **pMatrix2** directly. Given an affine function, we will use the spectral radius of the linear part to evaluate the contractivity. This gives exactly the similarity ratio of a similarity transformation and is roughly the largest scaling factor for a general affine transformation. Here is the computation of the similarity dimension for the sets  $A$  and  $B$ . Note that they should both have the same dimension, since each one contains a self-similar copy of the other.

```

matrices = Map[First,
  Transpose[newCurvesDigraph], {3}] // N;
eigenMatrix = matrices /.
  {{a_Real, b_Real}, {c_Real, d_Real}} →
  Max[Abs /@ Eigenvalues[{a, b}, {c, d}]];
sMatrix = Apply[Plus, eigenMatrix^s, {2}];
eigenvalues = Eigenvalues[sMatrix];
spectralRadius0 = Max[Chop[N[eigenvalues /. s → 0]]];
spectralRadius = Select[eigenvalues,
  ((# /. s → 0) == spectralRadius0) &][[1]];
approximateDim = Chop[s /.
  FindRoot[spectralRadius == 1, {s, 1}]]

1.24711

```

Note that the **DigraphFractals** package contains the function **ComputeDimension** which performs this task.

```

ComputeDimension[newCurvesDigraph]

1.24711

```

We now compute the Perron numbers for the matrix corresponding to this dimension, recalling that these form an eigenvector corresponding to the eigenvalue 1.

```

dimMatrix = sMatrix /. s → approximateDim;
Eigensystem[dimMatrix]

{{1., -0.12398}, {{0.918145, 0.396244}, {-0.647324, 0.762215}}}

perronNumbers = %[[2]][[1]]

{0.918145, 0.396244}

```

Recall that **pMatrix1** determines which node to move to from the current node. This is exactly where we want to use equation 2, which looks like the matrix  $A(s)$  times the eigenvector consisting of the Perron numbers. Thus we use **Inner** to form the equation, but we replace **Plus** with **List** so that we may use **FoldList** to get the result into the form which works with our algorithm.

```

pMatrix1 = Inner[Times, dimMatrix,
  perronNumbers, List] / perronNumbers;
pMatrix1 = FoldList[Plus, 0, #] & /@ pMatrix1;

```

To obtain **pMatrix2**, we simply normalize the correct **eigenMatrix**.

```

pMatrix2 = eigenMatrix /. s → approximateDim;
pMatrix2 = Map[pMatrixNormalizer, pMatrix2, {2}];

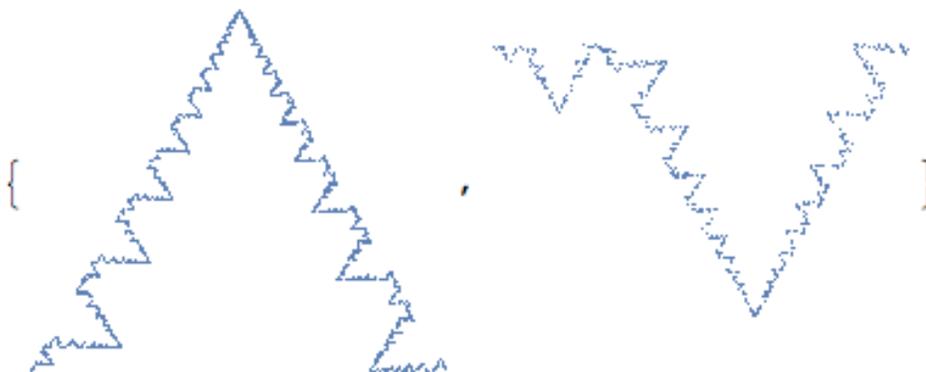
```

With **pMatrix1** and **pMatrix2** in hand, the algorithm may now proceed as before.

```

v = 1; v1 = pMatrix1[[v]]; v2 = pMatrix2[[v]];
currentPoint = {0, 0}; numPoints = 3000;
points = Table[{} , {Length[newCurvesDigraph]};
Do[{
  {i, j} = choose[v1, v2];
  currentPoint = digraphFuncs[[i, v, j]][currentPoint];
  points[[i]] = {points[[i]], currentPoint};
  v = i; v1 = pMatrix1[[v]]; v2 = pMatrix2[[v]];
}, {numPoints}];
points = Partition[Flatten[#, 2] & /@ points;
ListPlot[#,
  PlotStyle → {AbsolutePointSize[0.5]},
  AspectRatio → Automatic,
  Axes → False] & /@ points

```



This is a major improvement. A close look at these images, however, indicates that the first set is being traced out more heavily than the second. This is exactly the case, as may be deduced by investigating the Perron numbers.

```
perronNumbers
{0.918145, 0.396244}
```

These numbers indicate the relative "weights" of the sets, so we can see why the first is heavier. In fact, if we normalize these numbers taking the dimension into account, we should get the exact weights.

```
perronNumbers ^ approximateDim /
Plus @@ (perronNumbers ^ approximateDim)
{0.740387, 0.259613}
```

The following computation indicates that this has worked very well.

```
(Length /@ points) / numPoints // N
{0.745, 0.255}
```

This technique for choosing the probabilities is encoded in the package function **ComputePMatrix**, which goes further and combines the **pMatrix1** and **pMatrix2** obtained above into one **pMatrix**. Here is the **pMatrix** for our sets.

```
ComputePMatrix[newCurvesDigraph]
{{{0.698534}, {0.411257, 0.411257}}, {{0.301466}, {0.177486}}}
```

## More Examples

Our stochastic algorithm is encapsulated in the package function **ShowDigraphFractalsStochastic**. In this section, we illustrate the use of this function. The option **PMatrix** may be used to specify the probabilities for the algorithm and **PMatrix**  $\rightarrow$  **Automatic** generates the probabilities automatically. Here is the package description of the function **ShowDigraphFractalsStochastic**.

```
? ShowDigraphFractalsStochastic
```

```
ShowDigraphFractalsStochast[digraph_,numPoints_]
generates numPoints points approximating the digraph
Self Similar sets defined by digraph using a stochastic
algorithm The digraph is represented by a matrix of lists of
affine functions written in the form {{{a1,a2},{b1,b2},{x0,y0}}.
```

## ■ The Mauldin–Williams curves

Our first example is the digraph IFS for curves called the Mauldin–Williams curves in [Edgar 1990]. This example shows that our algorithm works on a digraph with more than two nodes.

```

a1 = {{{1/2, 0}, {0, -1/2}}, {0, 0}};
a2 = {{{1/2, 0}, {0, -1/2}}, {1/2, 0}};
α = ArcCos[17/18]; β = ArcCos[1/6];
b1 = {(1/3) rotate[-β], {Cos[α], Sin[α]}};
b2 = {rotate[α], {0, 0}};
c1 = {{{1/4, 0}, {0, -1/4}}, {0, 0}};
c2 = {{{3/4, 0}, {0, -3/4}}, {1/4, 0}};
MWCurves = {{{a1}, {a2}, {}},
             {{b1}, {}, {b2}},
             {{}, {c1}, {c2}}};

```

```
ShowDigraphFractalsStochastic[MWCurves, 6000]
```



## ■ A self-similar spiral

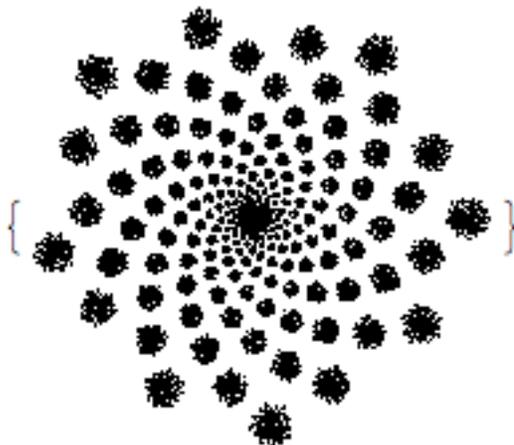
For our next example, we generate a self-similar spiral. This example shows that our technique works for a one-node digraph. The rotation in the transformation **a1** involves the golden ratio so that the resulting image has a double spiral appearance: 13 spirals in the clockwise direction and 21 in the counterclockwise direction. For a discussion on how the double spiral arises and the connection with the golden ratio and Fibonacci numbers, see chapter 9 of [Stewart 1995].

```

goldenAngle = 2 π (2 - GoldenRatio);
a1 = {.985 rotate[goldenAngle], {0., 0.}};
a2 = {{{.1, 0}, {0, .1}}, {1, 0.}};
spiral = {{{a1, a2}}};

ShowDigraphFractalsStochastic[spiral, 20 000]

```

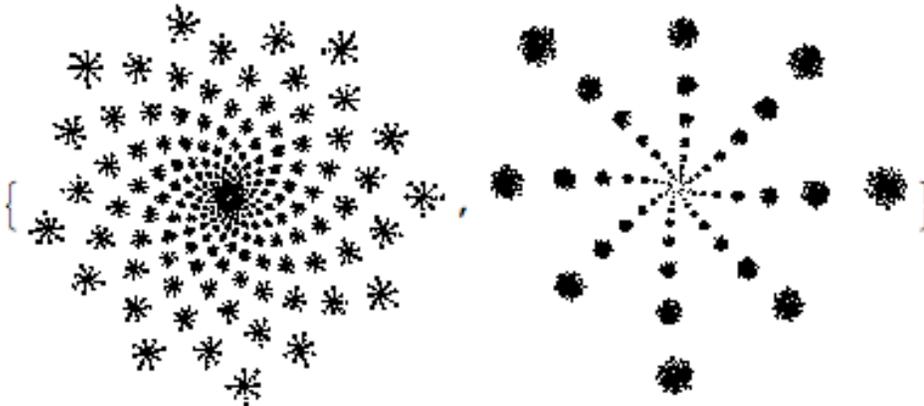


## ■ Enhancing the self-similar spiral

There are a few variations on the double spiral theme. Here is an example of two digraph self-similar spirals related to our last spiral. Note that for the second spiral below we change the angle of rotation just slightly. The resulting effect on the spiral is dramatic in the sense that the double spiral appearance is completely gone.

```
a1 = {.985 rotate[goldenAngle], {0., 0.}};
b1 = {.95 rotate[goldenAngle - 0.05], {0., 0.}};
a2 = b2 = {{{.1, 0}, {0, .1}}, {1, 0.}};
spirals = {{{a1}, {a2}}, {{b2}, {b1}}};

ShowDigraphFractalsStochastic[spirals, 30 000]
```



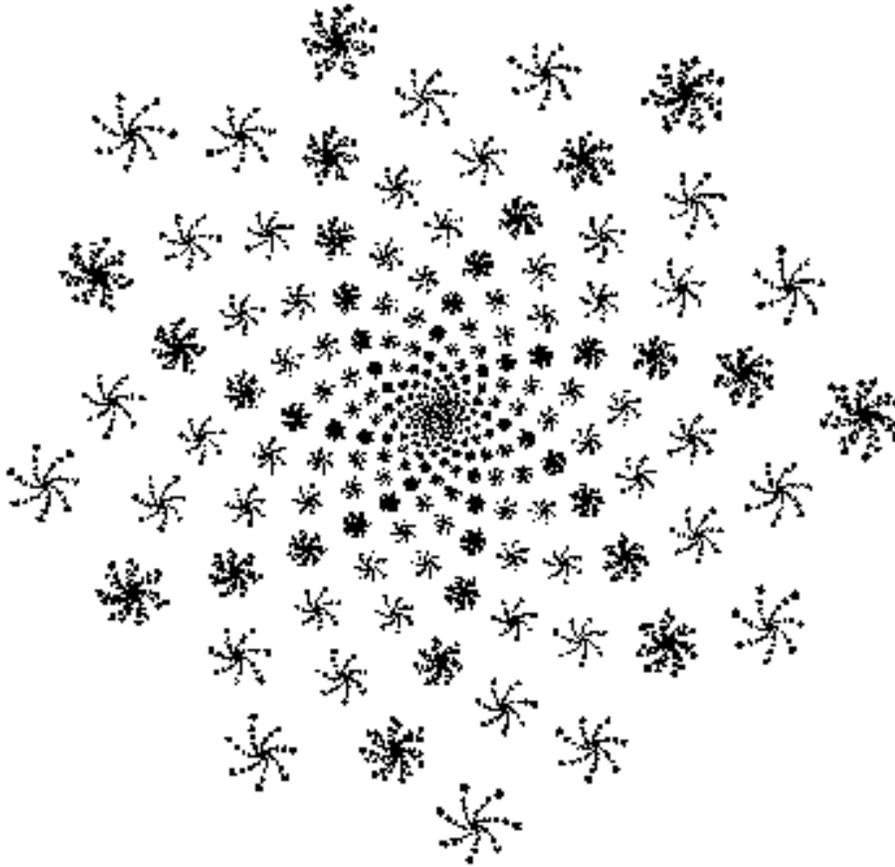
For our final example, we highlight every third spiral of the 21 spirals in the counterclockwise direction.

```
s1 = {.985^3 rotate[3 goldenAngle], {0, 0}};
s2 = {{{.1, 0}, {0, .1}}, {1, 0}};
d1 = {{{.1, 0}, {0, .1}}, .985 rotate[goldenAngle].{1, 0}};
d2 = {{{.1, 0}, {0, .1}}, .985^2 rotate[2 goldenAngle].{1, 0}};
d3 = s1;
intertwinedSpirals = {{{s1}, {s2}}, {{d1, d2}, {d3}}};
ShowDigraphFractalsStochastic[
  intertwinedSpirals, 30 000]
```



We can see why the name **intertwinedSpiral** was used as follows.

Show [%]



An image very similar to this appears on the cover of [Falconer 1997].

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## About the Author

Mark McClure is an assistant professor of mathematics at the University of North Carolina at Asheville. He received his Ph.D. in mathematics from the Ohio State University under the direction of Gerald Edgar. His

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