MML - Review for Exam 2

We will have our second exam this Friday, February 21. This review sheet is genuinely meant to help you succeed on that exam.

Generally, I will expect *solutions* to the problems, as opposed to just answers. So, for example, if the answer to an optimization problem is $y = 5$, then the solution will consist of a clear explanation with correctly written supporting computations indicating *why* the answer is $y = 5$.

The problems

1. Let U and V be subsets of $\mathbb{R}^2$ defined by

   $$U = \{\langle x, y\rangle \in \mathbb{R}^2 : 2x + 3y = 0\}$$

   and

   $$V = \{\langle x, y\rangle \in \mathbb{R}^2 : 2x + 3y = 5\}.$$

   Which of these is a vector space? For the one that is not, explain clearly why.

2. Suppose that the matrix A and its reduced row echelon form R are

   $$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 5 & 7 & 9 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

   a. Give a complete description of the column space of A.
   b. Give a complete description of the null space of A.
   c. Give a complete description of the range of A.

3. Show that the null space of a linear transformation is closed under linear combinations.

4. Let

   $$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

   a. Row reduce the augmented matrix $[A|I]$.

b. What is A^{-1}?

5. Suppose that A and B are invertible matrices of the same size. Show that $(AB)^{-1} = B^{-1}A^{-1}$.

6. Find a value of t such that

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 2t \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

are perpendicular.

7. Let $\langle \cdot, \cdot \rangle$ denote the inner product on $\mathbb{R}^2$ defined by

$$\langle \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T, \begin{bmatrix} y_1 & y_2 \end{bmatrix}^T \rangle = 2x_1y_1 + 3x_2y_2.$$

Also let $\mathbf{u} = \begin{bmatrix} 1 & 2 \end{bmatrix}^T$ and $\mathbf{v} = \begin{bmatrix} -2 & 1 \end{bmatrix}^T$.

a. Compute $\langle \begin{bmatrix} 1 & 2 \end{bmatrix}^T, \begin{bmatrix} -2 & 1 \end{bmatrix}^T \rangle$.
b. Are **u** and **v** orthogonal with respect to this inner product.
c. Show that this inner product is distributive over vector addition.

8. Find the projection $\text{proj}_{\mathbf{b}}\mathbf{x}$ of the vector **x** onto **b**, where

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

9. Find the least squares solution to the overdetermined linear system

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$