Calc III - Review for exam I

The first exam will be next Thursday, February 6. We will spend some time discussing a few of these problems in class on Tuesday but you should work them all out to the best of your ability prior to that. Understanding the problems on this sheet will help you greatly on the exam.

If you think that one of these problems has a mistake, you should try to find the mistake and fix it. Generally, I do not discuss exams problems during the exam, even if there is a mistake in the problem.

$$L\vec{c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 5 & 1 & 1 & 0 \\ -3 & -3 & -3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

1. Sketch the curve $(x-1)^2 + \frac{z+2}{4}^2 = 4$ in the appropriate coordinate plane. What does the graph of this equation look like in xyz-space?
2. Write down the equation of a sphere that is centered at the point $(3, 2, 1)$ and that has radius 9.
3. What object is described by the equation $x^2 + y^2 + z^2 = 2z$?
4. An object moves according to the parametrization $\boldsymbol{p}(t) = \langle t + \cos(4t), -\sin(4t)\rangle$.
   (a) Describe the motion determined by $\boldsymbol{p}(t)$.
   (b) Write down a parametrization of the line tangent to the path at the point when $t = \pi/3$.
5. Let $\boldsymbol{v} = \langle 1, -2, 3\rangle$ and let $\boldsymbol{w} = \langle -1, 1, 3\rangle$.
   (a) Compute $\boldsymbol{v} \times \boldsymbol{w}$ and show that it is perpendicular to $\boldsymbol{v}$, in this particular case.
   (b) Find $\text{proj}_{\boldsymbol{v}}\boldsymbol{w}$, the vector projection of $\boldsymbol{w}$ onto $\boldsymbol{v}$.
6. The position of a projectile is given by $\vec{p}(t) = \langle 4t, 64t - 16t^2\rangle$.
   (a) Find the velocity and acceleration of the projectile.
   (b) Assuming the projectile stops when it returns to height zero, what is the horizontal distance traveled by the projectile?
7. The position of a particle is given by $\vec{r}(t) = (-\cos(3t), \sin(3t))$.
   (a) Sketch the path traced out by the particle. Describe the direction of motion.
   (b) Find the velocity $\vec{v}(t)$ of the particle as a function of time.
   (c) Show that the velocity vector $\vec{v}(t)$ is always perpendicular to the position vector $\vec{r}(t)$.
8. Let $\vec{a} = \langle 1, -2, 4\rangle$ and $\vec{b} = \langle 2, 5, 3\rangle$. Decompose $\vec{a}$ into the sum of a vector parallel to $\vec{b}$ and a vector perpendicular to $\vec{b}$.

9. Let $\boldsymbol{v} = \langle 2, -1, 3 \rangle$ and let $\boldsymbol{w} = \langle -1, 2, -3 \rangle$.
   (a) Compute $3\boldsymbol{v}$, $-\frac{1}{2}\boldsymbol{v}$, $\boldsymbol{v} + \boldsymbol{w}$, $\boldsymbol{v} - \boldsymbol{w}$, $\boldsymbol{v} \cdot \boldsymbol{w}$, and $\boldsymbol{v} \times \boldsymbol{w}$.
   (b) What does the sign of $\boldsymbol{v} \cdot \boldsymbol{w}$ tell you about the geometric relationship between $\boldsymbol{v}$ and $\boldsymbol{w}$?
   (c) What is the exact angle between $\boldsymbol{v}$ and $\boldsymbol{w}$?
   (d) What is the geometric relationship between $\boldsymbol{v}$, $\boldsymbol{w}$, and $\boldsymbol{v} \times \boldsymbol{w}$?
   (e) Find $\mathbf{proj}_{\boldsymbol{v}}\boldsymbol{w}$.
10. Find a value of t so that $\left\langle 1, t, t^2 \right\rangle$ is perpendicular to $\langle 1, 1, 1 \rangle$.
11. Possible proofs
    (a) Let $\boldsymbol{v} = \langle x, y \rangle$ and let $\boldsymbol{w} = r\boldsymbol{v}$. Show that $\|\boldsymbol{w}\| = r\,\|\boldsymbol{v}\|$.
    (b) Prove that the two-dimensional dot product is commutative and distributive over vector addition. Is it associative? Why?
    (c) Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ be three dimensional vectors. Prove that $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$.
12. Describe the motions determined by the following parametrized paths.
    Note: Some are 2D and some are 3D.
    (a) $\boldsymbol{p}(t) = \langle 1, 1, 1 \rangle + \langle 2, 3, -1 \rangle t$
    (b) $\boldsymbol{p}(t) = \langle 1, 1, 1 \rangle + \langle 2, 3, -1 \rangle e^t$
    (c) $\boldsymbol{p}(t) = \langle \cos(2t), -\sin(2t) \rangle$
    (d) $\boldsymbol{p}(t) = \langle \cos(2t), -\sin(2t), 10 \rangle$
    (e) $\boldsymbol{p}(t) = \langle \cos(2t), -\sin(2t) + 10 \rangle$
    (f) $\boldsymbol{p}(t) = \langle \cos(2t), -\sin(2t), t \rangle$
    (g) $\boldsymbol{p}(t) = \langle \cos(2t), -\sin(2t) + t \rangle$
    (h) $\boldsymbol{p}(t) = \langle t\cos(2t), t\sin(2t) \rangle$
13. Find the equation of a plane containing the points $(1, -2, 3)$, $(-1, 2, 2)$, and $(1, 1, 2)$ or explain why no such plane exists.
14. Find the area of the triangle in space whose vertices are $(1, -2, 3)$, $(-1, 2, 2)$, and $(1, 1, 2)$.
15. Find a parametrization of the line through the point $(1, 2, 3)$ and perpendicular to the plane $2x - y + 3z = 6$.
16. Find the equation of a plane containing the points $(1, 2, 2)$, $(-1, 3, 2)$, and $(1, 1, 2)$ or explain why no such plane exists.
17. Let $p(t) = (3 + 2t, 1 - t, 4 + t)$ and let $q(t) = (-2 + 3t, 2t, 2 + t)$ be the parameterizations of two lines.
    (a) Find the point of intersection of the two lines.
    (b) Find an equation of the plane that contains the two lines.
18. Let $p(t) = (3 + 2t, 1 - t, 4 + t)$ and let $q(t) = (-2 + 3t, 2t, 4 + t)$ be the paramertrizations of two lines. Find the distance between those lines.

19. Parametrize the intersection between the sphere of radius two centered at the point $(1, 1, 1)$ with the xy plane.

20. Match the groovy function below with the groovy picture in figure 1.

    (a) $p(t) = t(\sin(t), \cos(t))$

    (b) $p(t) = t(\cos(t), \sin(t))$

    (c) $p(t) = (t, t) + (\cos(4t), \sin(4t))$

    (d) $p(t) = (\cos(5t), \sin(4t))$

    (e) $p(t) = (\cos(t), \sin(t)) + (\cos(20t), \sin(20t))/5$

    (f) $p(t) = (\cos(t), \sin(t)) + (\cos(7t), \sin(7t))/2 + (\cos(-17(t + \pi/2)), \sin(-17(t + \pi/2)))/3$

    (g) $p(t) = (0, \cos(t), \sin(t))$

    (h) $p(t) = (t, \cos(t), \sin(t))$

    (i) $p(t) = (t, t\cos(t), t\sin(t))$

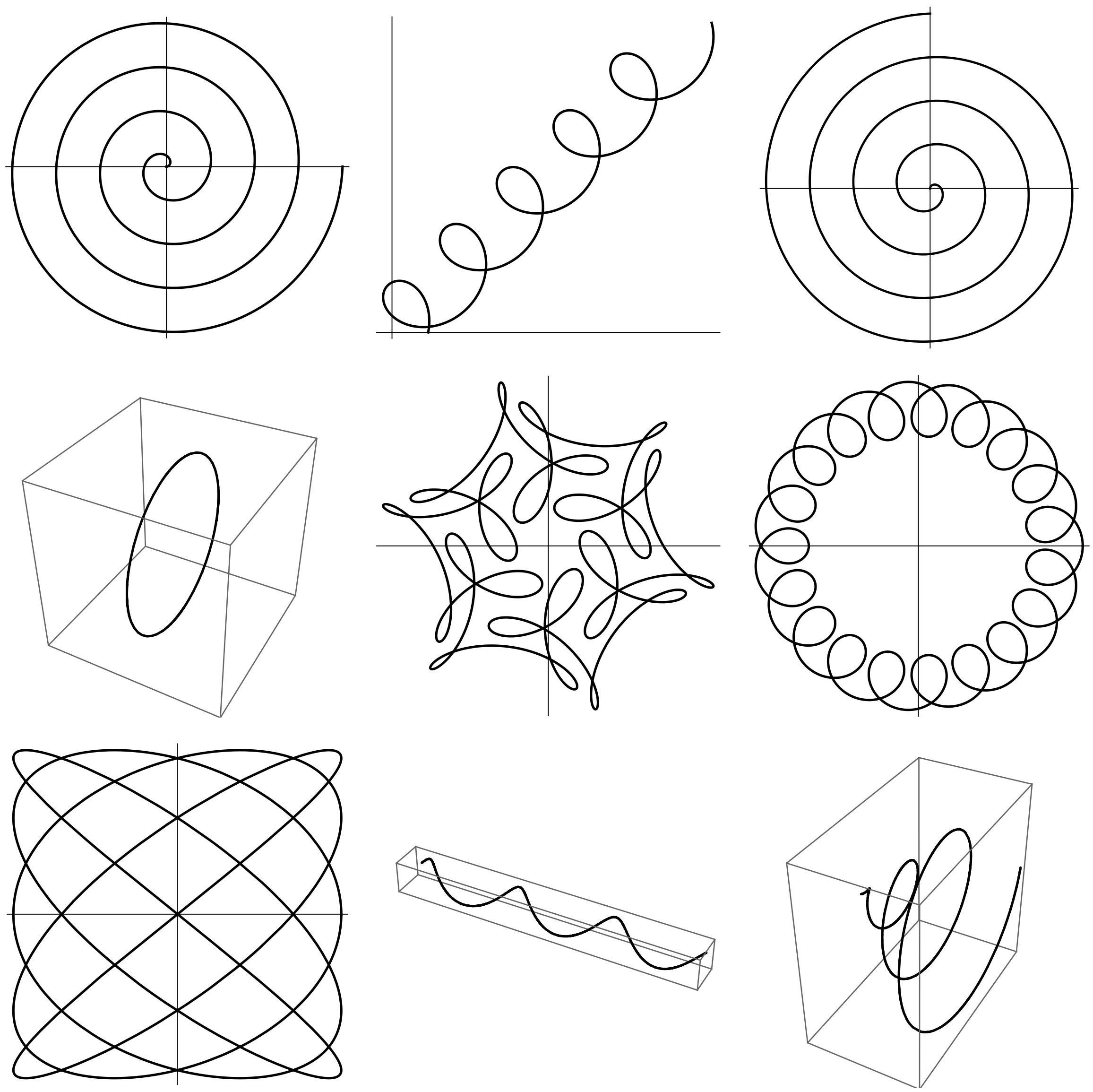

Figure 1: Some parametric paths