Complex Dynamics

Problems for Quiz/Exam 2

1. Consider the iteration of the polynomial $f(z) = z^7$.
   (a) Show that zero is a super-attractive fixed point of f.
   (b) Show that $e^{\pi i/3}$ is a repulsive fixed point of f.
   (c) Explain clearly why f displays sensitive dependence on initial conditions on the unit circle.
2. State the general escape criterion for complex polynomials.
3. Let $f(z) = 2z^5 - z^4 + 3z^3 - 8z^2 + z - 1$. What is the escape radius for f guaranteed by the polynomial escape criterion?
4. Let $f(z) = z^3 - \frac{3}{2}az^2 + b$.
   (a) Show that the critical points of f are a and zero.
   (b) Write down a system of equations that a and b should satisfy in order for f to have super-attractive orbits of periods 1, and 2.
   (c) What is the escape radius for f guaranteed by the polynomial escape criterion?
5. Let $f(z) = z^4 - \frac{4}{3}(a + b)z^3 + 2abz^2 + c$.
   (a) Show that the critical points of f are a, b, and zero.
   (b) Write down a system of equations that a, b, and c should satisfy in order for f to have super-attractive orbits of periods 1, 2, and 3.
   (c) What is the escape radius for f guaranteed by the polynomial escape criterion?
6. Let $f(z) = z^2 + c$ and let $F(z) = 1/f(1/z)$. Show that zero is a super-attractive fixed point of F.
7. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is differentiable on $\mathbb{C}$ and that $z_0 \in \mathbb{C}$ is an attractive fixed point of f. Show that there is an $r > 0$ such that $f^n(z) \to z_0$ for all z such that $|z - z_0| < r$. (This is Lemma 5.1.1.)

8. State the linearization theorem (Theorem 5.1.2) for attractive fixed points.
9. State the Leau-Fatou Flower Theorem (Theorem 5.4.4).
10. Let $f(z) = \sin(z^3)$.
    (a) Show that the origin is a parabolic point for f.
    (b) Find attractive and repelling direction vectors for f at zero.