Finding Limits Analytically

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In this handout, we explore analytic techniques to compute limits.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x)+g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 1 Basic Limit Properties

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Sums/Differences: $\lim_{x\to c}(f(x) \pm g(x)) = L \pm K$
4. Scalar Multiples: $\lim_{x\to c} b \cdot f(x) = bL$
5. Products: $\lim_{x\to c} f(x) \cdot g(x) = LK$
6. Quotients: $\lim_{x\to c} f(x)/g(x) = L/K$, $(K \neq 0)$
7. Powers: $\lim_{x\to c} f(x)^n = L^n$
8. Roots: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
9. Compositions: Adjust our previously given limit situation to:

   $$\lim_{x\to c} f(x) = L, \lim_{x\to L} g(x) = K \text{ and } g(L) = K.$$

   Then $\lim_{x\to c} g(f(x)) = K$.

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L.

We apply the theorem to an example.

Example 1 Using basic limit properties

Notes:

Let

$$\lim_{x\to 2} f(x) = 2, \quad \lim_{x\to 2} g(x) = 3 \quad \text{and} \quad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim_{x\to 2} \big(f(x) + g(x)\big)$
2. $\lim_{x\to 2} \big(5f(x) + g(x)^2\big)$
3. $\lim_{x\to 2} p(x)$

Solution

1. Using the Sum/Difference rule, we know that $\lim_{x\to 2} \big(f(x) + g(x)\big) = 2 + 3 = 5$.

2. Using the Scalar Multiple and Sum/Difference rules, we find that $\lim_{x\to 2} \big(5f(x) + g(x)^2\big) = 5 \cdot 2 + 3^2 = 19$.

3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}
\lim_{x\to 2} p(x) &= \lim_{x\to 2} (3x^2 - 5x + 7) \\
&= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7 \\
&= 3 \cdot 2^2 - 5 \cdot 2 + 7 \\
&= 9
\end{aligned}$$

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 1. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

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Theorem 2 Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim_{x\to c} p(x) = p(c)$
2. $\lim_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, where $q(c) \neq 0$.

Example 2 Finding a limit of a rational function

Using Theorem 2, find

$$\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3}.$$

Solution Using Theorem 2, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3} &= \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

The section could have been titled "Using Known Limits to Find Unknown Limits." By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x) \leq g(x) \leq h(x).$$

If f and h have the same limit at c, and g is always "squeezed" between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

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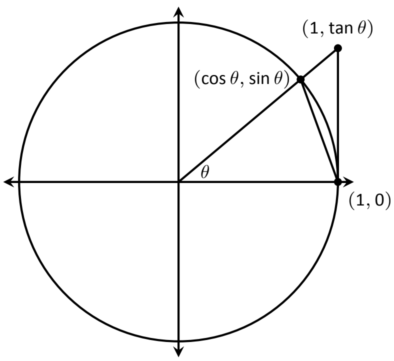

Figure 1: The unit circle and related triangles.

Theorem 3 Squeeze Theorem

Let f, g and h be functions on an open interval I containing c such that for all x in I,

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to "squeeze" the given function of which you are trying to evaluate a limit. However, that is generally the only place work is necessary; the theorem makes the "evaluating the limit part" very simple.

We use the Squeeze Theorem in the following example to finally prove that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Example 3 Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Solution We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 1. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$. Later we will show that we can also consider $\theta \leq 0$.)

Figure 1 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that

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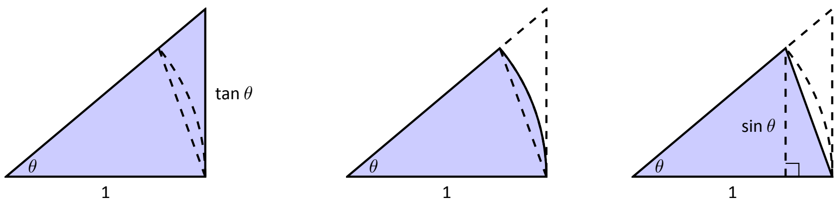

$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply all terms by $\dfrac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.)

Now take limits.

$$\lim_{\theta\to 0} \cos\theta \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

$$\cos 0 \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq 1$$

Clearly this means that $\displaystyle\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

Two notes about the previous example are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin x$ are approaching 0, but the *ratio*

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of x and $\sin x$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin x$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 4 Special Limits

1. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$
2. $\lim\limits_{x\to 0} \dfrac{\cos x - 1}{x} = 0$
3. $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = e$
4. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos x$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos x$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

Our final theorem for this section will be motivated by the following example.

Example 4 Using algebra to evaluate a limit

Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Solution We begin by attempting to apply Theorem **??** and sub-

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stituting 1 for x in the quotient. This gives:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\frac{1^2-1}{1-1}=\text{“}\ \frac{0}{0}\ \text{”},$$

and indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure 2, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x=1$, but for all other x,

$$\frac{x^2-1}{x-1}=\frac{(x-1)(x+1)}{x-1}=\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}=x+1.$$

Clearly $\lim\limits_{x\to 1} x+1=2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except $x=1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2.$$

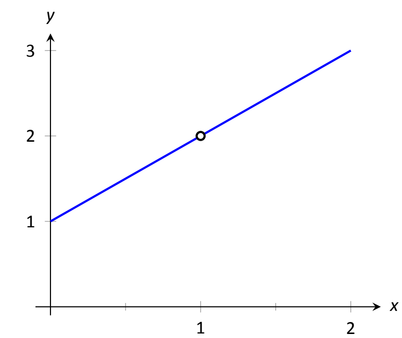

Figure 2: Graphing f in Example 4 to understand a limit.

The key to the above example is that the functions $y=(x^2-1)/(x-1)$ and $y=x+1$ are identical except at $x=1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Theorem 5 Limits of Functions Equal At All But One Point

Let $g(x)=f(x)$ for all x in an open interval, except possibly at c, and let $\lim\limits_{x\to c} g(x)=L$ for some real number L. Then

$$\lim_{x\to c} f(x)=L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit

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$\lim_{x\to c}\frac{g(x)}{f(x)}$ returns "0/0", then $(x-c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this term out, cancel, then apply Theorem 5. We demonstrate this once more.

Example 5 **Evaluating a limit using Theorem 5**

Evaluate $\lim_{x\to 3}\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}$.

Solution We begin by applying Theorem **??** and substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}=\frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}.$$

We can cancel the $(x-3)$ terms as long as $x\neq 3$. Using Theorem 5 we conclude:

$$\begin{aligned}\lim_{x\to 3}\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}&=\lim_{x\to 3}\frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}\\&=\lim_{x\to 3}\frac{(x^2+x-2)}{(2x^2+9x-5)}\\&=\frac{10}{40}=\frac{1}{4}.\end{aligned}$$

We end this section by revisiting a limit first seen in Section **??**, a limit of a difference quotient. Let $f(x)=-1.5x^2+11.5x$; we approximated the limit $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}\approx 8.5$. We formally evaluate this limit in the following example.

Example 6 **Evaluating the limit of a difference quotient**

Let $f(x)=-1.5x^2+11.5x$; find $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$.

Solution Since f is a polynomial, our first attempt should be to employ Theorem **??** and substitute 0 for h. However, we see that this gives us "0/0." Knowing that we have a rational function hints that some

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algebra will help. Consider the following steps:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{-1.5(1+h)^2+11.5(1+h)-\big(-1.5(1)^2+11.5(1)\big)}{h}\\
&= \lim_{h\to 0}\frac{-1.5(1+2h+h^2)+11.5+11.5h-10}{h}\\
&= \lim_{h\to 0}\frac{-1.5h^2+8.5h}{h}\\
&= \lim_{h\to 0}\frac{h(-1.5h+8.5)}{h}\\
&= \lim_{h\to 0}(-1.5h+8.5) \quad \big(\text{using Theorem 5, as } h\neq 0\big)\\
&= 8.5 \quad \big(\text{using Theorem } \textbf{??}\big)
\end{aligned}$$

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem **??**; it states that many functions that we use regularly behave in a very nice, predictable way. In the next section we give a name to this nice behavior; we label such functions as *continuous.* Defining that term will require us to look again at what a limit is and what causes limits to not exist.

Notes:

Exercises 0.0

In Exercises 1 – 8, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim_{x\to 9} f(x) = 6$, $\lim_{x\to 6} f(x) = 9$, $f(9) = 6$
- $\lim_{x\to 9} g(x) = 3$, $\lim_{x\to 6} g(x) = 3$, $g(6) = 9$

1. $\lim_{x\to 9} (f(x) + g(x))$
2. $\lim_{x\to 9} (3f(x)/g(x))$
3. $\lim_{x\to 9} \left(\frac{f(x) - 2g(x)}{g(x)}\right)$
4. $\lim_{x\to 6} \left(\frac{f(x)}{3 - g(x)}\right)$
5. $\lim_{x\to 9} g\big(f(x)\big)$
6. $\lim_{x\to 6} f\big(g(x)\big)$
7. $\lim_{x\to 6} g\big(f(f(x))\big)$
8. $\lim_{x\to 6} f(x)g(x) - f^2(x) + g^2(x)$

In Exercises 9 – 12, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim_{x\to 1} f(x) = 2$, $\lim_{x\to 10} f(x) = 1$, $f(1) = 1/5$
- $\lim_{x\to 1} g(x) = 0$, $\lim_{x\to 10} g(x) = \pi$, $g(10) = \pi$

9. $\lim_{x\to 1} f(x)^{g(x)}$
10. $\lim_{x\to 10} \cos\big(g(x)\big)$
11. $\lim_{x\to 1} f(x)g(x)$
12. $\lim_{x\to 1} g\big(5f(x)\big)$

In Exercises 13 – 27, evaluate the given limit.

13. $\lim_{x\to 3} x^2 - 3x + 7$
14. $\lim_{x\to \pi} \left(\frac{x-3}{x-5}\right)^7$
15. $\lim_{x\to \pi/4} \cos x \sin x$
16. $\lim_{x\to 0} \ln x$
17. $\lim_{x\to 3} 4^{x^3 - 8x}$
18. $\lim_{x\to \pi/6} \csc x$
19. $\lim_{x\to 0} \ln(1 + x)$
20. $\lim_{x\to \pi} \frac{x^2 + 3x + 5}{5x^2 - 2x - 3}$
21. $\lim_{x\to \pi} \frac{3x + 1}{1 - x}$
22. $\lim_{x\to 6} \frac{x^2 - 4x - 12}{x^2 - 13x + 42}$
23. $\lim_{x\to 0} \frac{x^2 + 2x}{x^2 - 2x}$
24. $\lim_{x\to 2} \frac{x^2 + 6x - 16}{x^2 - 3x + 2}$
25. $\lim_{x\to 2} \frac{x^2 - 10x + 16}{x^2 - x - 2}$
26. $\lim_{x\to -2} \frac{x^2 - 5x - 14}{x^2 + 10x + 16}$

27. $\lim_{x\to -1} \dfrac{x^2+9x+8}{x^2-6x-7}$

Use the Squeeze Theorem in Exercises 28 – 31, where appropriate, to evaluate the given limit.

28. $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$

29. $\lim_{x\to 0} \sin x\cos\left(\dfrac{1}{x^2}\right)$

30. $\lim_{x\to 1} f(x)$, where $3x-2 \leq f(x) \leq x^3$.

31. $\lim_{x\to 3} f(x)$, where $6x-9 \leq f(x) \leq x^2$.

Exercises 32 – 36 challenge your understanding of limits but can be evaluated using the knowledge gained in this section.

32. $\lim_{x\to 0} \dfrac{\sin 3x}{x}$

33. $\lim_{x\to 0} \dfrac{\sin 5x}{8x}$

34. $\lim_{x\to 0} \dfrac{\ln(1+x)}{x}$

35. $\lim_{x\to 0} \dfrac{\sin x}{x}$, where x is measured in degrees, not radians.

36. Let $f(x) = 0$ and $g(x) = \dfrac{x}{x}$.

   (a) Show why $\lim_{x\to 2} f(x) = 0$.

   (b) Show why $\lim_{x\to 0} g(x) = 1$.

   (c) Show why $\lim_{x\to 2} g\big(f(x)\big)$ does not exist.

   (d) Show why the answer to part (c) does not violate the Composition Rule of Theorem 1.