Complex iteration

Introductory examples

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May 23, 2017

We begin our foray into complex dynamics by taking a look at some examples. While we won't dive heavily into the theory just yet, we'll assume that some of the results we learned from real iteration are still applicable. For example, the definitions of fixed points and periodic orbits are still clearly applicable and we'll assume that our classification as attractive, repulsive, or neutral is still applicable. Also, our definition of conjugation and it's consequences were stated at quite a general level. Thus, we'll feel free to conjugate a function to simplify things when convenient.

1 Affine functions

The dynamics of affine iteration is quite simple. When we iterate $f(z) = az+b$ there just a few possibilities, assuming $a \neq 0$:

1. $|a| < 1$: In this case, f has a unique, attractive fixed point, say z_0. If z_1 is *any* complex number, then iteration of f from z_1 will converge towards z_0 at an exponential rate, i.e. like $a^n \to 0$. If a is not a real number, then this convergence will spiral in to z_0.

2. $|a| > 1$: Again, f has a unique fixed point, say z_0, that is now repelling. If z_1 is *any* complex number other than z_0, then iteration of f from z_1 will diverge to ∞ like $a^n \to \infty$. If a is not a real number, then this convergence will spiral out.

3. $|a| = 1$: Assuming that $a \neq 1$, application of f represents pure rotation about some point. If $a = 1$ and $b \neq 0$, then f is a shift. In the final case that $a = 1$ and $b = 0$, f is the identity function.

These observations characterize the dynamics of affine iteration completely. As a result, we will restrict further attention to more complicated maps.

2 z^2: The unit circle

The simplest quadratic, obviously, is $f(z) = z^2$. In fact, it's easy to obtain a closed form expression for the n^{th} iterate of f, namely: $f^n(z) = z^{2^n}$. In spite of this formula, iteration of f displays tremendously interesting dynamics indicative of much of what is to come. As we'll see, in fact, the dynamics of f naturally decomposes the complex plane into two regions: a stable region and an unstable region. A similar type of decomposition is possible for *all* polynomials and, even, for all rational functions. We'll come to call the stable region the *Fatou set* and the unstable region the *Julia set*.

We'll try to paint a static picture of the dynamics here but there is a dynamic and interactive illustration of the actual orbits of f here: https://goo.gl/xBvFeo.

2.1 The stable region

Let's suppose that $|z_0| < 1$, i.e. z_0 is in the interior of the unit circle in the complex plane. Then, since $f^n(z_0) = z_0^{2^n}$, it's easy to see that the iterates of z_0 converge to zero. In addition, the argument of the iterate doubles with each application of f. As a result, there is a spiral effect as well.

If $|z_0| > 1$, i.e. z_0 is in the exterior of the unit circle in the complex plane, then the iterates spiral out away from the circle and towards infinity. The dynamics of f is shown for several choices of z_0 in figure 2.1.

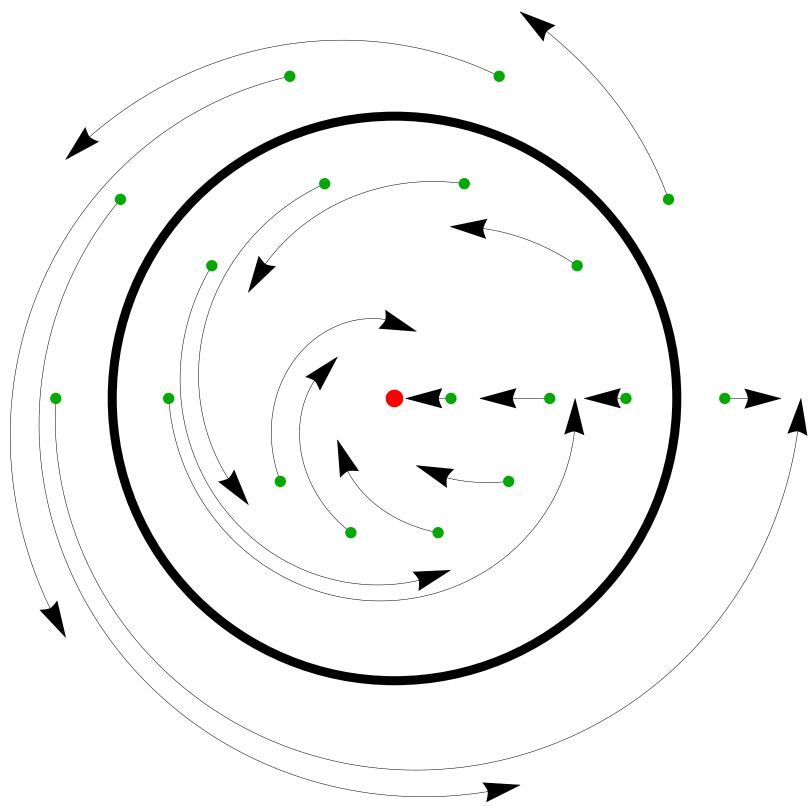

Figure 2.1: The dynamics of z^2

Let F denote the complement of the unit circle. Thus, F is an open set consisting of two, disjoint parts - the interior of the unit circle and the exterior. The dynamics of f on F are stable in a concrete sense. Suppose that $z_0 \in F$. Then, there is a disk D centered at z_0 whose radius is small enough that $D \subset F$. This means that D lies entirely within the interior of the unit circle or within the exterior. Either way, the long term behavior of every point in D is the same as the long term behavior of z_0. This is exactly what we mean by stability in F: changing the initial input a little bit does *not* change the long term behavior. This is exactly the opposite of sensitive dependence on initial conditions.

2.2 The unstable region

We now turn our attention to the dynamical behavior of f on the unit circle and we'll see that it's as complicated as it can be. Let's denote the unit circle by J. Our first order of business is to show that f displays sensitive dependence on J, so let $z_0 \in J$ and let D be an open disk centered at z_0. Then, no matter how small the radius of D, it will always contain points that are in the interior of J and points in the exterior of J. Thus, there are points arbitrarily close to z_0 whose orbits converge to zero and other points arbitrarily close to z_0 whose orbits diverge to ∞. That's pretty sensitive!

What about the behavior of f right on J? As it turns out, f is conjugate to the doubling map d and, therefore, chaotic on J. Recall that d is defined on $H = \{x : 0 \leq x < 1\}$ by

$$d(x) = 2x \bmod 1.$$

In fact, it's trivial to show that a conjugacy from f to d is given by the natural parametrization of the unit circle $x \to e^{2\pi i x}$. This is one of those wonderful moments in mathematics where a lot of prior work comes together to yield an important result quite easily!

3 $z^2 - 2$: A line segment

We now let $f(z) = z^2 - 2$. From our work in real iteration, we already know that $f : [-2, 2] \to [-2, 2]$ and that f is chaotic on that interval. On the complement, $\mathbb{C} \setminus [-2, 2]$, it turns out that f is stable. In fact, the iterates of f diverge to ∞ for every initial point $z_0 \in \mathbb{C} \setminus [-2, 2]$.

To see this, let $F = \mathbb{C} \setminus [-2, 2]$ and let E denote the exterior of the unit disk, i.e. $E = \mathbb{C} \setminus \{z : |z| \leq 1\}$. We'll show that the action of f on F is conjugate to the action of the squaring function $g(z) = z^2$ on E.

The conjugacy function will be $\varphi(z) = z + 1/z$. The geometric action of φ is shown in figure 3.1 We first show that φ maps the boundary of E (the unit circle) to the boundary of F (the interval $[-2, 2]$). Of course, the points on the unit circle are exactly those points of the form e^{it} for some $t \in [0, 2\pi)$. Thus, we compute

$$\begin{aligned}
\varphi(e^{it}) &= e^{it} + e^{-it} \\
&= (\cos(t) + i\sin(t)) + (\cos(t) - i\sin(t)) \\
&= 2\cos(t).
\end{aligned}$$

The expression $2\cos(t)$ traces out the interval $[-2, 2]$ (twice, in fact) as t ranges through $[0, 2\pi)$ as claimed.

We next show that φ is one-to-one on E. To this end, suppose that

$$z + \frac{1}{z} = w + \frac{1}{w}.$$

Then,

$$z - w = \frac{1}{w} - \frac{1}{z} = \frac{z - w}{wz}.$$

Assuming that $z \neq w$, we can divide off the numerators and then take reciprocals to obtain $wz = 1$. Thus, if z is a point in the exterior of the unit disk, there is exactly one other point w such that $\varphi(w) = \varphi(z)$, namely the reciprocal of z which lies in the interior of the unit disk.

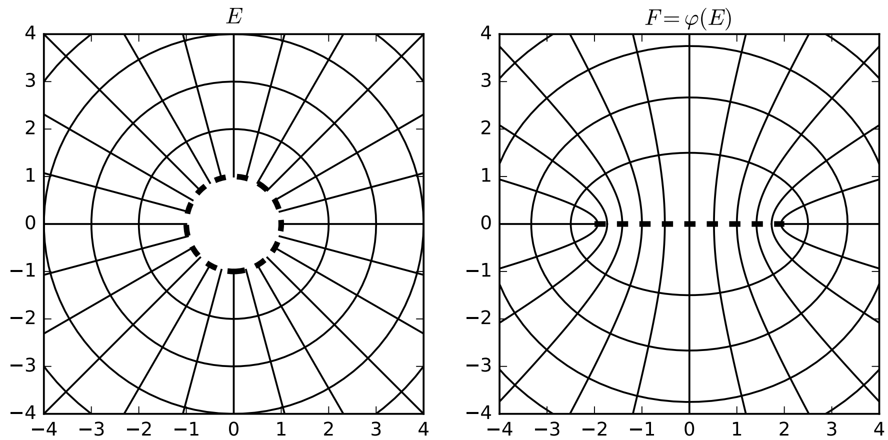

Figure 3.1: The conjugacy from F to E

4 $z^2 - 1$: A fractal

The dynamic behavior illustrated by the previous two examples is quite typical for the iteration of a polynomial on $\mathbb{C}$. The plane decomposes into two disjoint regions - a stable region and an unstable region. By the very characterization of stability, the stable region is an open set and the unstable region is closed. Points in the stable region converge to some attractive behavior while the behavior is chaotic on the unstable region.

The geometry of the unstable region is typically much more complicated than the geometry in the previous two examples. Let's explore the function $f(z) = z^2 - 1$ experimentally to see what other types of behavior can arise.

First, let's give a little thought to the types of behavior that might arise. It seems reasonable to suspect that, if $|z|$ is large, then *any* quadratic will behave something like z^2. Thus, we expect that, once an iterate is large enough, its orbit will escape to ∞. Let's try to prove this a little more rigorously. Suppose that $|z_0| > 2$ and write $|z_0| = 2 + \varepsilon$, where ε must be some positive number. Then,

$$\begin{aligned} |f(z_0)| &= |z_0^2 - 1| \geq \big||z_0^2| - |1|\big| = |z_0|^2 - 1 \\ &\geq |z_0^2| - |z_0| = |z_0|(|z_0| - 1) = |z_0|(1 + \varepsilon). \end{aligned}$$

By induction, $|f^n(z_0)| \geq |z_0|(1 + \varepsilon)^n \to \infty$. Thus, once an orbit exceeds 2 in absolute value, it *must* escape to ∞

On the other hand, there are certainly plenty of orbits that are bounded because, as we know, the points 0 and -1 form a super attractive orbit of period 2. Thus, there should be a little disk about each of those points containing points that are attracted to that orbit. A reasonable question is, how close must we be to one of those points to guarantee convergence to that orbit? To address this, let

$$F(z) = f(f(z)) = (z^2 - 1)^2 - 1 = z^4 - 2z^2.$$

Note that 0 and -1 are both super-attractive fixed points of F. We can be confident that an iterate z_k of F will converge to one of those fixed points if it's close to the fixed point *and* $|F'(z_k)| < 1$.

Taking all this into account, let's describe an experimental procedure to explore the dynamics of f. First, we define a rectangular region in the complex plane. Specifically, let R denote the square with lower left corner at the point $-2.2-2.2i$ and with upper right corner $2.2+2.2i$. Break that square up into pixels of dimension, say, 1000×1000 - each corresponding to a complex number z_0. For each of those pixels, iterate from z_0 until one of three things happens.

1. The iterate z_k exceeds 2 in absolute value
2. $|F'(z_k)| < 1$ and $\text{Re}(z_k) < 0.618$
3. We're tired

We then shade the result according to how many times we've iterated. The result is shown in figure 4.1.

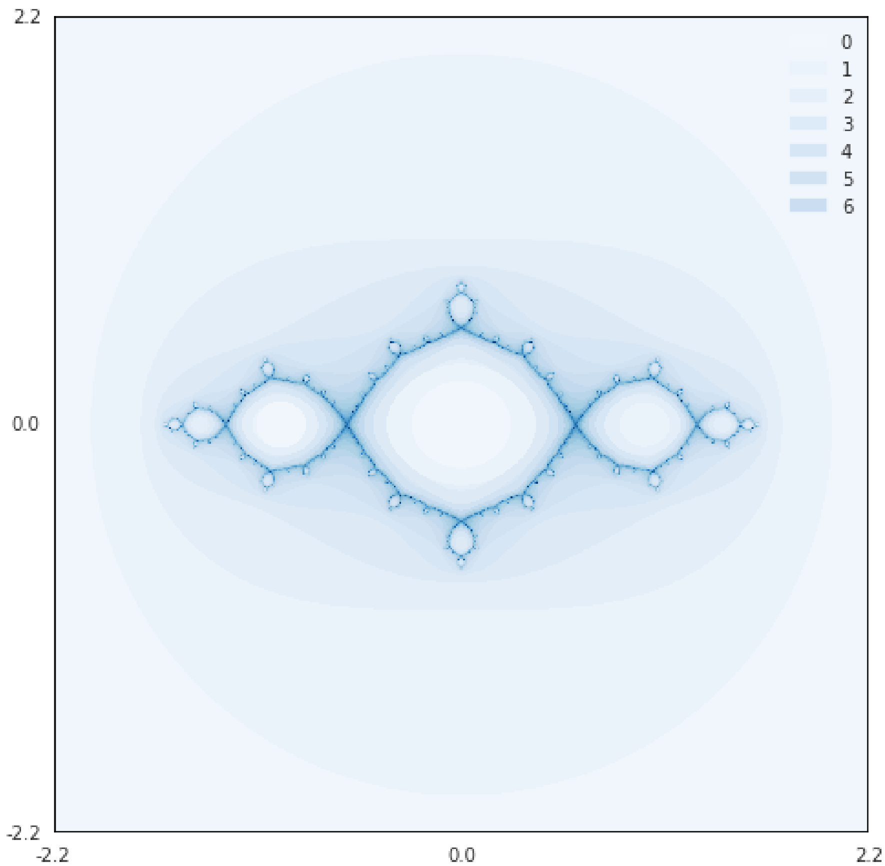

Figure 4.1: Convergence into the stable region for $f(z) = z^2 - 1$

An interactive orbit generator for this function is included on this webpage: https://goo.gl/04o9AY.

5 A polynomial algorithm

Although we've only looked at a few examples, it seems apparent (for polynomials, at least), that the orbit of any initial seed that is large enough will diverge to ∞. This is not at all hard to prove.

Theorem 5.1 (Polynomial escape criterion)**.** *Let*

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0.$$

Then, the orbit of z_0 diverges to ∞ whenever $|z_0|$ exceeds

$$R = \max\left(2|a_n|, 2\frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}\right).$$

The number R is called the escape radius *for the polynomial.*

Proof. Suppose that $z_0 \in \mathbb{C}$ satisfies $|z_0| > 2|a_n|$ and

$$|z_0| > 2\frac{|a_{n-1}| + \cdots + |a_0|}{|a_n|}.$$

Then,

$$\begin{aligned} |z_1| = |f(z_0)| &= |a_n z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_0| \\ &= |a_n z_0^n| \left|1 + \frac{a_{n-1}}{a_n z_0} + \cdots + \frac{a_0}{a_n z_0^n}\right|. \end{aligned}$$

Now, our condition on z_0 implies that

$$\begin{aligned} S \equiv \left|\frac{a_{n-1}}{a_n z_0} + \cdots + \frac{a_0}{a_n z_0^n}\right| &\leq \left|\frac{a_{n-1}}{a_n z_0}\right| + \cdots + \left|\frac{a_0}{a_n z_0^n}\right| \\ \leq \left|\frac{a_{n-1}}{a_n z_0}\right| + \cdots + \left|\frac{a_0}{a_n z_0}\right| &= \frac{|a_{n-1}| + \cdots + |a_0|}{|a_n z_0|} < \frac{1}{2}. \end{aligned}$$

Thus, the triangle inequality yields

$$|1| = |1 + S + (-S)| \leq |1 + S| + |-S| = |1 + S| + |S|$$

so that $|1 + S| \geq 1 - |S| \geq 1/2$. Applying this to our original inequality for z_1, we obtain

$$\begin{aligned} |z_1| &> |a_n z_0^n| \frac{1}{2} \geq |a_n||z_0||z_0| \frac{1}{2} \\ &> 2|z_0| \frac{1}{2} = z_0. \end{aligned}$$

Now, since $|z_1| > |z_0|$ we can write $|z_1| = \lambda|z_0|$ for some $\lambda > 1$. Furthermore, z_1 now also satisfies $z_1 > R$ so that, by induction, $|z_n| \geq \lambda^n |z_0|$. As a result, $z_n \to \infty$ as $n \to \infty$. □

It's easy to strengthen this result. Suppose we iterate from an initial seed z_0 that may be less than the escape radius. If any iterate ever does exceed the escape radius in absolute value, then the orbit will escape. This yields the following corollary.

Corollary 5.2. *Suppose we iterate a polynomial from an initial seed z_0. Then, there are two mutually exclusive possibilities for the orbit of z_0*

1. *The orbit stays bounded by the escape radius, or*
2. *the orbit diverges to ∞.*

Theorem 5.1 ensures that case 2 happens for all polynomials. It's also easy to see that there are lots of points whose orbits stay bounded. Any fixed point, yields an orbit that stays bounded. Of course, there are always fixed points since the equation $f(x) = x$ yields a polynomial that always has at least one *complex* solution. For that matter, periodic orbits are also bounded and $f^n(x) = x$ yields *lots* of those when n is large.

Definition 5.3. Let $p : \mathbb{C} \to \mathbb{C}$ be a polynomial.

1. The *filled Julia set* of p is the set of all complex numbers whose orbits remain bounded under iteration of p.

2. The *Julia set* of p is the boundary of the filled Julia set.

3. The complement of the filled Julia set or, equivalently, the set of all points that diverge to ∞ is called the *basin of attraction of* ∞.

We will denote the filled Julia set of p by K_p and the Julia set of p by J_p.

We should point out that there are other characterizations of the Julia set that work more generally than the definition we give here, simply because other classes of functions might not have an "escape radius". When we redefine the Julia set in those other contexts, it will also be our responsibility to prove that it is consistent with this first definition.

There are a few relatively easy observations that we can make about these sets. Suppose that for some initial seed z_0 and integer n, we have $|p^n(z_0)| > R$. Then, by continuity, the same will be true for points sufficiently close to z_0. Thus, the basin of attraction of p is an *open* set. As a consequence, the filled Julia set K_p is a closed set. The boundary of a set is, by definition, its closure minus its interior. Thus the Julia set itself is also closed and non-empty. In addition, any neighborhood of J_p contains points whose orbits diverge to ∞ and points that stay bounded. Thus, the dynamics of p near J_p display sensitive dependence on initial conditions.

Finally, we note that corollary 5.2 yields a nice algorithm to generate a filled Julia set.

Algorithm 5.4 (The escape time algorithm (for polynomial Julia sets))**.**

1. *Choose some rectangular region in the complex plane bound on the lower left by, say, z_{min} and on the upper right by z_{max}.*

2. *Partition this region into large number of rows and columns. We'll call the intersection of a row and column a "pixel", which corresponds to a complex number.*

3. *For each pixel, iterate p until one of two things happens:*

   (a) *We exceed the escape radius in absolute value, in which case we shade the pixel according to how many iterates it took to escape.*

   (b) *We exceed some pre-specified maximum number of iterations, in which case we color the pixel black.*

In spite of the simplicity of this section, many of the ideas here are similar to more complicated situations and we will see several variations of the escape time algorithm later. The remainder of our examples rely on application of this algorithm

6 $z^2 - 0.123 + 0.745i$: Douady's rabbit

Douady's rabbit is the Julia set of $p(z) \approx z^2 - 0.123 + 0.745i$. More precisely, it's the Julia set of $f_c(z) = z^2 + c$ where $c \approx -0.123 + 0.745i$ is chosen so that $f_c^3(0) = 0$. As a result, p has a super-attractive orbit of period 3.

Douady's rabbit is shown in figure 6.1 with the critical orbit shown in red.

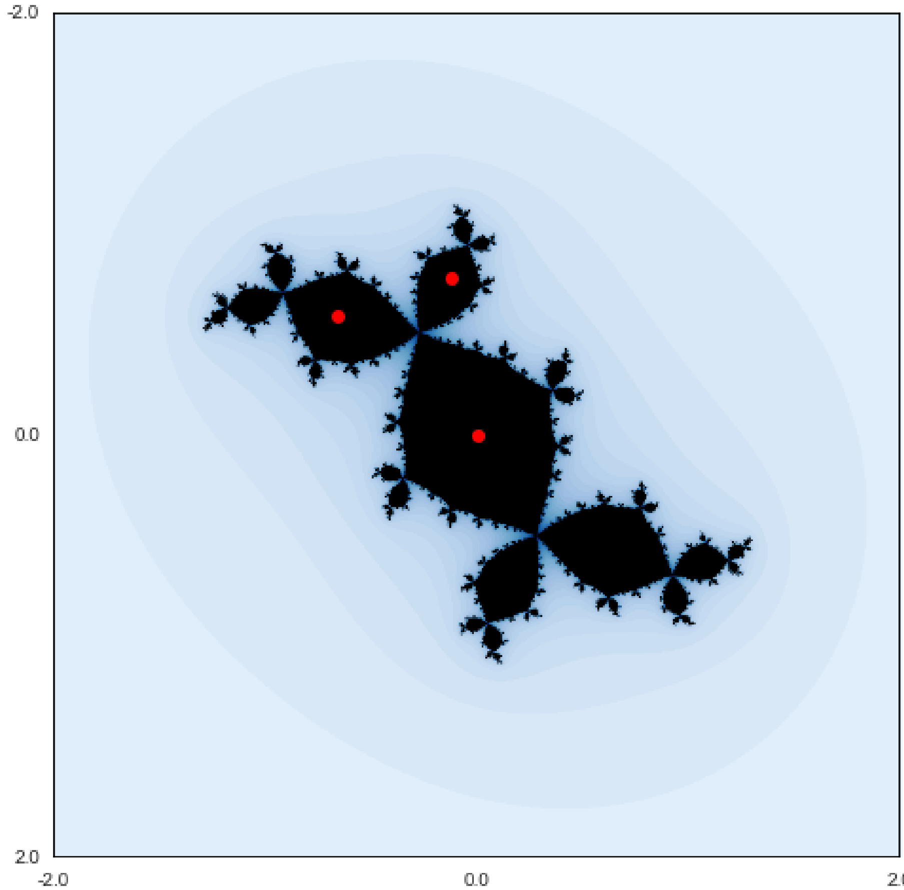

Figure 6.1: Douady's rabbit

An interactive orbit generator for this function is included on this webpage: https://goo.gl/04o9AY.

7 A cubic with two attractive orbits

Consider the cubic polynomial

$$p(z) = -z^3 + (2.41154 - 0.133695i)z^2 - (0.090706 - 0.27145i),$$

whose filled Julia set is shown in figure 7.1. This polynomial has attractive orbits of periods both two and three. A major point here is that more complicated behavior can arise, when the degree of the polynomial increases or, more generally, as the function becomes more complicated.

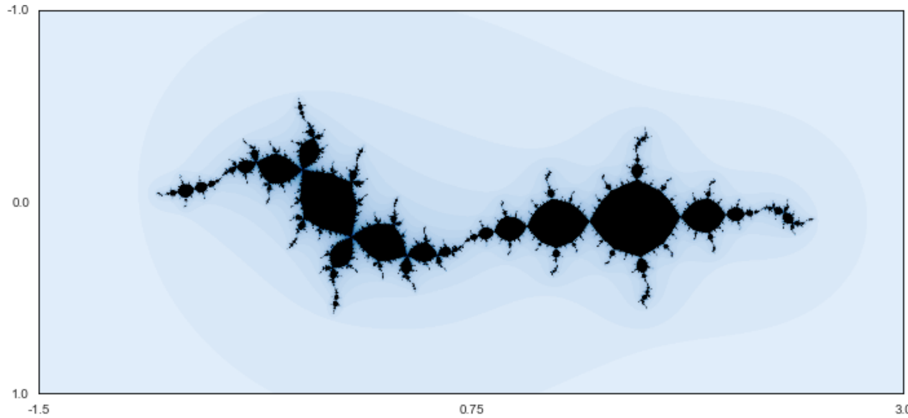

Figure 7.1: A cubic with two attractive orbits

An interactive orbit generator for this function is included on this webpage: https://goo.gl/04o9AY.

8 $z + z^5$: Neutrality

We now consider the polynomial $f(z) = z + z^5$. It's a polynomial, right? So let's just apply our algorithm! The result is shown in figure 8.1.

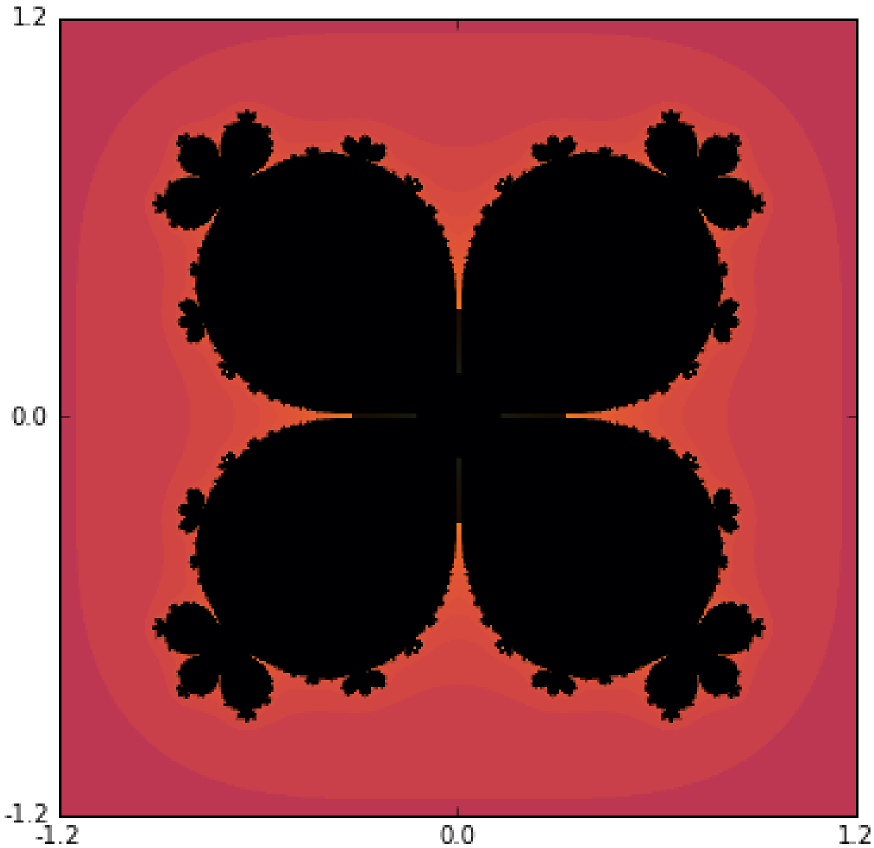

Figure 8.1: The escape algorithm for $f(z) = z + z^5$

Well, it seems clear that something's going on near the origin but it's not so clear exactly what that is. It looks like maybe there's a solid disk contained in the black region and centered at the origin. Maybe, though, there are four separate lobes that meet at an articulation point (also called a "pinchy thing") there.

Ultimately, a solid description of the behavior near zero depends on the dynamical behavior there. Thus, to address this, first note that zero is a fixed point. Of course, if zero was an *attractive* fixed point, then there should be an open disk about it contained entirely in the black region. But zero isn't attractive; it's neutral, as a simple calculation shows: $f'(z) = 1 + 5z^4$, thus $f'(0) = 1$. The behavior near a neutral fixed point is much more complicated than the behavior near an attractive point and there are a variety of possibilities. Claim 8.2 provides some description of the behavior in this particular example.

Claim 8.2. *Let $z_0(r,n) = re^{n\pi i/4}$. Then, if $|r| < 1$, the iterates of f from $z_0(r,n)$ move away from zero when n is even but move towards zero when n is odd.*

Proof. We simply compute

$$f(z_0(r,n)) = re^{n\pi i/4} + (re^{n\pi i/4})^5 = re^{n\pi i/4} + r^5 e^{5n\pi i/4}.$$

Note that we always get the initial point $z_0(r,n)$ plus a displacement $r^5 e^{5n\pi i/4}$. The $e^{5n\pi i/4}$ term represents the direction of the displacement. Now, if n is even, we may write $n = 2k$ for some integer k so that

$$e^{5n\pi i/4} = e^{10k\pi i/4} = e^{(8k+2k)\pi i/4} = e^{2k\pi i/4} = e^{n\pi i/4},$$

by the periodicity of the exponential function. Thus, the displacement is in the direction of the initial input and away from the origin. If, however, n is odd, we must write $n = 2k+1$ and then,

$$e^{5n\pi i/4} = e^{5(2k+1)\pi i/4} = e^{(8k+(2k+1)+4)\pi i/4} = e^{((2k+1)+4)\pi i/4} = e^{(n+4)\pi i/4}.$$

Thus, the direction of the displacement is now the original direction *plus* and additional $4\pi/4 = \pi$, which points us in the opposite direction of the initial input. Furthermore, when $|r| < 1$, we have $|r^5| < |r|$. Thus, the magnitude of the displacement is smaller than the magnitude of the initial input so we step back towards the origin but not past it. ☐

Figure 8.3 illustrates the dynamics of f together with a good image of its Julia set. The blue points are the pre-images of the origin. The green arrows indicate directions along which points move towards the origin. The red arrows indicate directions along which points move away from the origin.

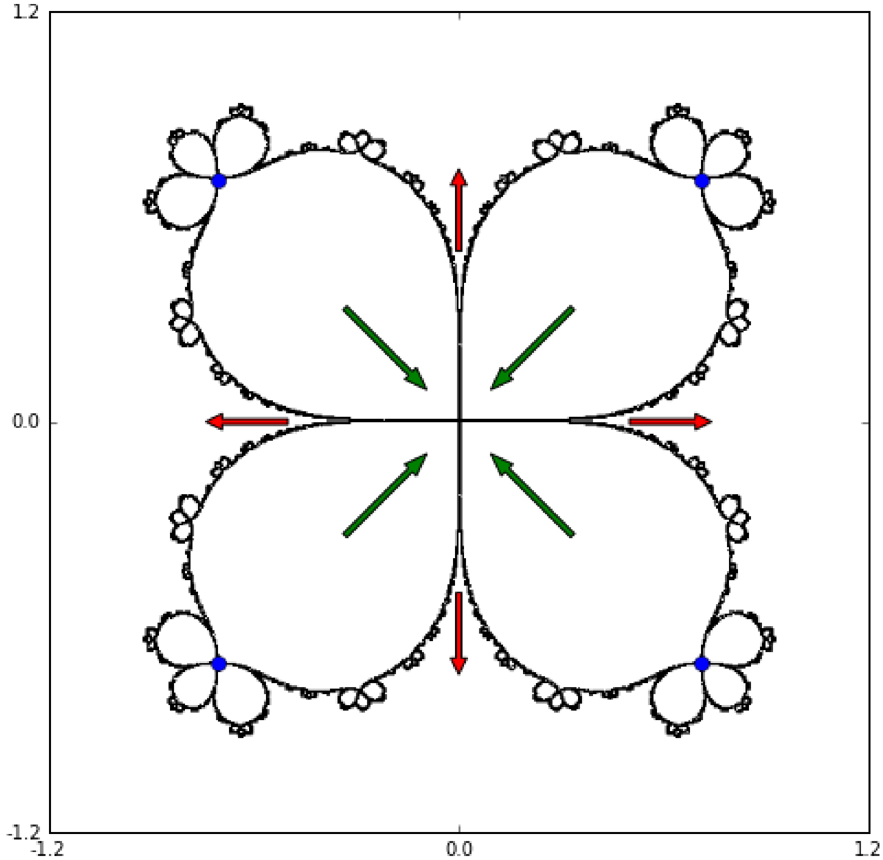

Figure 8.3: The dynamics of $f(z) = z + z^5$

In spite of our newly discovered analytic understanding of the dynamics of f, it's still reasonable to ask why our polynomial algorithm didn't do a better job at generating a picture. This is a polynomial after all, and we do expect orbits to escape rapidly. Why didn't the algorithm work well? Can we improve it by simply increasing maximum number of iterations we are willing to try? Figure 8.1 sets a max iteration count of 1000, which is already much higher than the other figures we've generated.

The issue is that the dynamics near zero move *very* slowly. So slowly, that increasing the iteration count won't really help. Let's try to develop a little mathematical machinery to analyze this. As we've seen, if we start with a point $z_0 > 0$ and we apply f, then we will generate another *real* number that's even larger. Thus, iteration of f from $z_0 > 0$ say n times generates an increasing list:

$$z_0 < z_1 < z_2 < \cdots < z_n.$$

Note that the number n could be written

$$n = \sum_{i=1}^{n} 1 = \sum_{i=0}^{n-1} \frac{1}{f(z_i) - z_i}(z_{i+1} - z_i),$$

because $z_{i+1} = f(z_i)$. While, this might seem like a crazy way to write n, the final sum is a left-handed Riemann sum for the function $F(z) = 1/(f(z) - z) = 1/z^5$ and is therefore an approximation to a definite integral of F. Furthermore, since F is decreasing, the left Riemann sum will yield an upper bound for the integral it approximates or, turning things around, the integral yields a lower bound for n.

To apply this, suppose we ask the question - how many iterates of f does it take to move a the point $z_0 = 1/100$ to the point $2/100$? A lower bound is provided by the integral

$$\int_{1/100}^{2/100} \frac{1}{z^5} dz = 23437500.$$

That's over 23 million iterates to move a just one point only a little bit! To be clear, a resolution of 1/100 is not particularly good for the types of images we are interested in, so simply increasing our iteration count is not a viable option.

The Julia set shown in figure 8.3 was generated using a completely different technique. Each of the four large lobes meeting at the origin is invariant under the action of f. Each of the small lobes that we see eventually maps onto one one of those four large invariant lobes and then stays there. Thus, we can iterate a modest number of times and classify the initial point according to which of the four quadrants it's landed in. If we shade the initial point according to that classification, we generate figure 8.4. If we then perform a boundary scan of that figure, we generate the Julia set shown in figure 8.3.

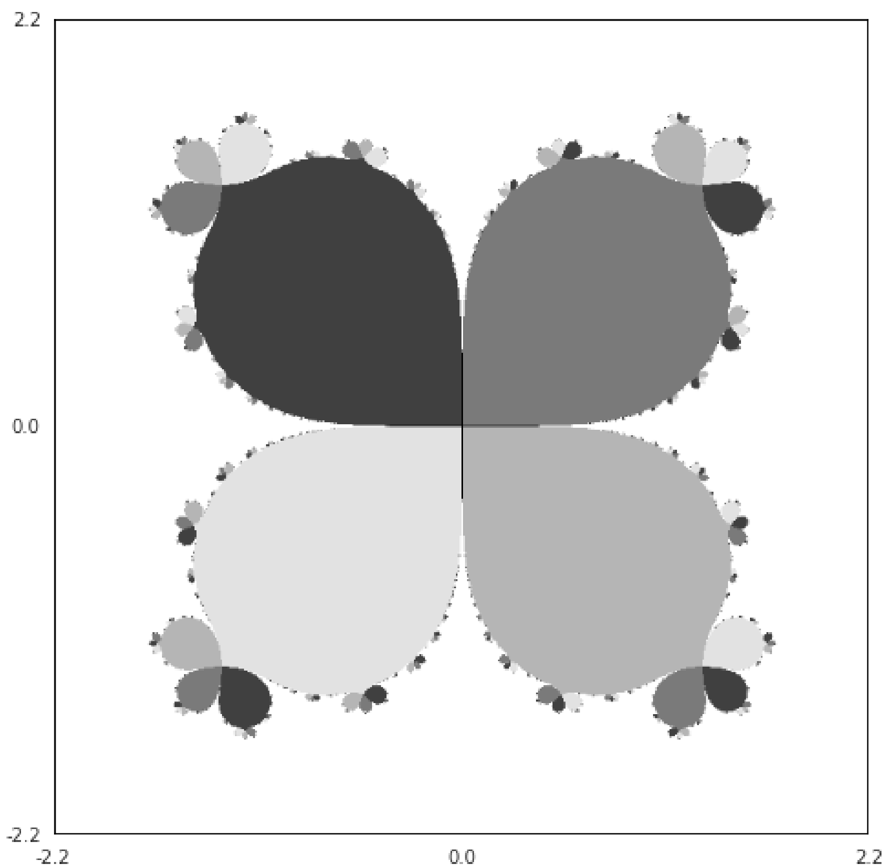

Figure 8.4: Classification according to which lobe z_0 land in

9 $e^x/3$: A hairy exponential

To this point, our functions have been polynomials. Polynomials are awesome! They always have fixed points and many other periodic orbits; they always critical points; and there's always a reasonable escape criterion. As we move away from polynomials and consider other types of functions, we lose most of that. However (as we'll prove in the next chapter), the behavior of fixed and periodic points (*if* they exist) remains consistent.

A function that is differentiable at every point in the complex plane that is *not* a polynomial is called an *entire* function. Perhaps the simplest example of such a function is the exponential function. In this section, we'll explore the dynamical behavior of $f(z) = e^z/3$. While plain old e^z might seem more natural, it lacks any attractive behavior. Our function f has exactly one real fixed point as a simple graph shows.

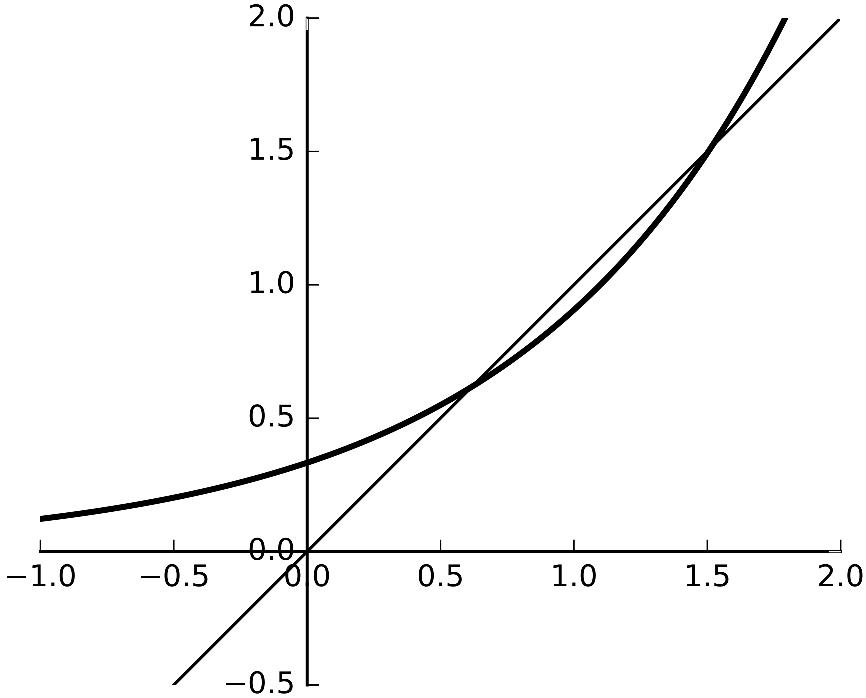

Figure 9.1: The graph of $f(z) = e^z/3$, together with the line $y = x$

Now, to understand the dynamical behavior of f, we can choose a region containing the fixed point, decompose it into pixels, and iterate from each pixel until we are either close to the fixed point or ... something else I guess. An exponential function grows quickly for large values of $\text{Re}(z)$. So it might make sense to bail out for, say, $\text{Re}(z) > 500$. The result is shown in figure 9.2.

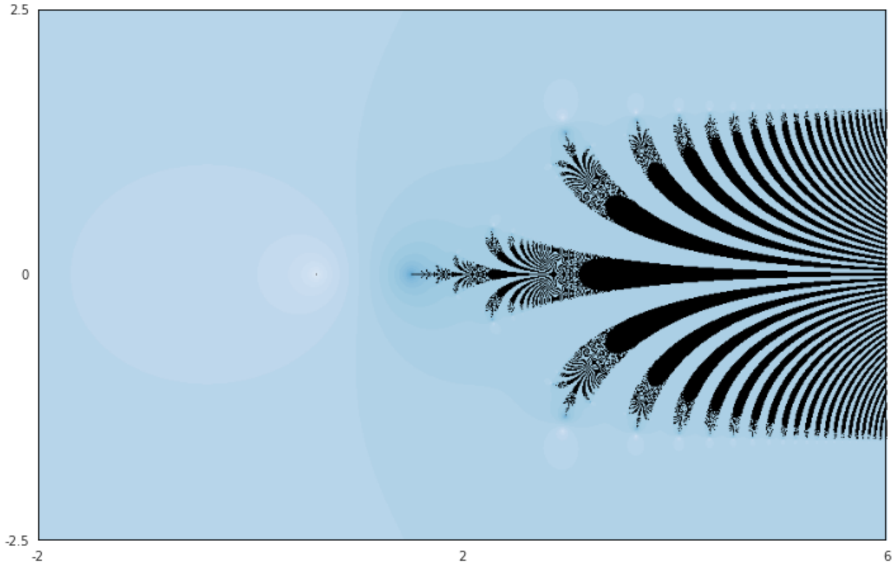

Figure 9.2: A basin of attraction for $f(z) = e^z/3$

10 $z/2 - 1/(2z)$: Newton's method applied to a quadratic

Suppose we apply Newton's method to the function $f(z) = z^2 + 1$. We get

$$N(z) = z - \frac{z^2+1}{2z} = \frac{z}{2} - \frac{1}{2z}.$$

It's easy to show that $\pm i$ are both super-attractive fixed points. It seems easy to believe that any point in the upper half plane should converge to i, while any point in the lower half plane should converge to $-i$ - and, that's exactly what happens. Hardly seems worth a picture. We'll do better - we'll *prove* it!

Claim 10.1. *Let $f(z) = z^2+1$ so that $N(z) = z/2 - 1/(2z)$ is the corresponding Newton's method iteration function. Let z_0 be an initial seed for iteration under N. If $Im(z_0) > 0$, then the orbit of z_0 converges to i; if $Im(z_0) > 0$, then the orbit of z_0 converges to $-i$. Furthermore, the restriction of N to the real line is conjugate to $g(z) = z^2$ on the unit circle and, thus, chaotic there.*

Proof. For the conjugacy function, we choose the Mobius transformation

$$\varphi(z) = \frac{z-i}{z+i}.$$

Note that $\varphi(\infty) = 1$, $\varphi(0) = -1$, and $\varphi(1) = -i$. Thus, the image of the real axis is exactly the unit circle. To show that φ conjugates g to N, we simply compute:

$$\varphi(N(z)) = \frac{\frac{z}{2} - \frac{1}{2z} - i}{\frac{z}{2} - \frac{1}{2z} + i} = \frac{2z^2 - 2 - 4iz}{2z^2 - 2 - 4iz}$$
$$= \frac{2(z-i)^2}{2(z+i)^2} = \left(\frac{z-i}{z+i}\right)^2 = g(\varphi(z)).$$

Furthermore, $\varphi(i) = 0$ and $\varphi(-i) = \infty$. Thus the super-attractive fixed point i for N corresponds to the super-attractive fixed point at the origin for

g and the top half of the plane maps to the interior of the unit circle. As a result, everything in the top half of the plane converges to i under iteration of N, just as everything in the interior of the unit circle converges to the origin under iteration of g. The correspondence between the bottom half of the plane and the exterior of the unit circle is similar. □

Claim 10.1 can be generalized. Let f be a complex quadratic with distinct roots z_1 and z_2 and let N be the corresponding Newton's method iteration function. It can be shown that the the line of points equidistant from z_1 and z_2 is invariant under N and the dynamics of N on that line are conjugate to $z \to z^2$ on the unit circle. Furthermore, the orbit of an initial seed off of that line will converge to the closer of the the two roots. This result was established independently by Ernst Schroder and Arthur Cayley in the 1870s. This is probably the first theorem of complex dynamics. Both authors explored the same idea for cubic polynomials and commented that there were significant difficulties. It was exactly this challenge that gave rise to complex dynamics as a field of study.

11 $z + \cot(z)$: Newton's method applied to the cosine

Suppose we apply Newton's method to the cosine. We should generate a function $n(z)$ with a super-attractive fixed point at each root of the cosine, i.e. at each odd multiple of $\pi/2$: $(2k+1)\pi/2$. Thus, we have infinitely many basins of attraction. If we apply our standard computational technique of iterating for each point in a grid - coloring the initial point according to which root we converged to and shading according to how long the convergence took, we get figure 11.1. If we zoom in near the origin, we generate figure 11.2.

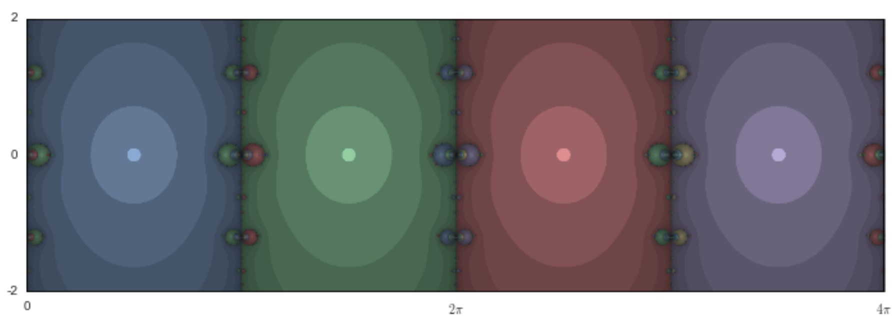

Figure 11.1: Basins of attraction for Newton's method applied to the cosine

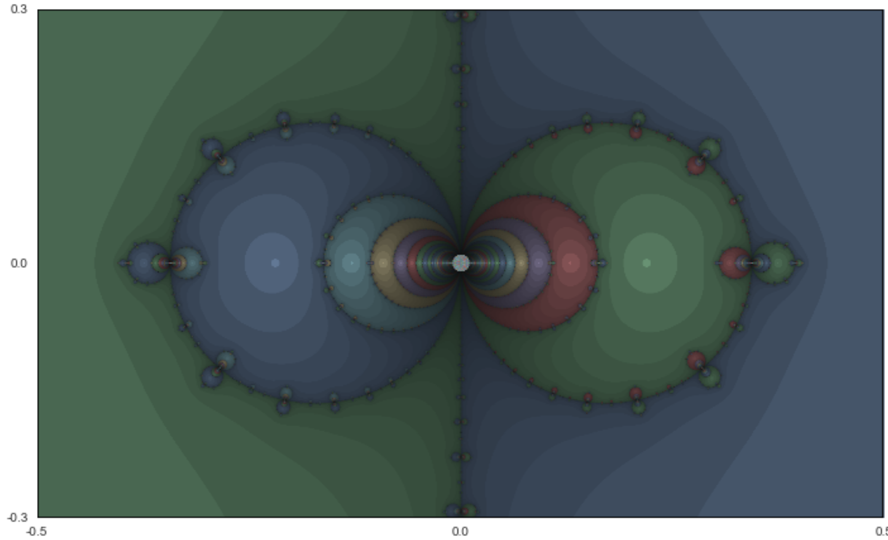

Figure 11.2: A zoom into figure 11.1

We can gain an understanding of the complication displayed in figure 11.2 by considering the graph of the cosine near zero. Since there is a maximum there, the tangent line procedure that determines which basin we land in is very sensitive to small changes.

12 Exercises

1. Consider iteration of the function $f(z) = z^3$.

(a) Show that zero is a super-attractive fixed point of f.

(b) Show that the orbit of z_0 tends to zero whenever $|z_0| < 1$ but diverges to ∞

(c) Explain precisely why f displays sensitive dependence on initial conditions.

(d) Compute the orbits of $e^{\pi i/3}$ and $e^{\pi i/4}$.

2. Let $f(z) = z^3 - z - 1$.

(a) Determine the fixed points of f and classify as attractive, repelling, or neutral.

(b) Plot the filled Julia set of f and indicate the locations of the the fixed points.

3. Let $f(z) = 2z^5 - z^4 + 3z^3 - 8z^2 + z - 1$. What is the escape radius guaranteed by theorem 5.1?

4. Show that $z_0 = 1$ is a neutral fixed point of $f(z) = e^{z-1}$.

5. In section 8, we examined the function $f(z) = z + z^5$ motivated, in part, by dissatisfaction with figure 8.1. Specifically, we asked - "Does the stable region contain the origin?" Using the family $f_r(z) = rz + z^5$, show that it's possible for a function to have a Julia set that qualitatively looks like figure 8.1, yet is stable at zero.

6. Consider iteration of the function $f(z) = z - z^3$. Note that zero is a neutral fixed point. Show that real points near zero move towards zero under iteration of f but that imaginary points near zero move away from zero under iteration of f.

7. Let $f(z) = z^2 + 1$ and let $N(z)$ be the corresponding Newton method iteration function. Show by direct computation that i is a super-attractive fixed point for N.