Real Analysis II - Review for exam I

The first exam will be this Wednesday, February 24. Here are some problems worth consideration.

1. Write down a careful statement of the following theorems.
   (a) Uniform convergence
   (b) Abel's theorem
   (c) Taylor's theorem with remainder
   (d) Weierstrass M-test
   (e) The upper and lower sums $U(f,P)$ and $L(f,P)$ of a bounded function with respect to a partition
   (f) The upper and lower sums $U(f)$ and $L(f)$ of a bounded function over an interval
   (g) The Riemann integral $\displaystyle\int_a^b f$ of a bounded function f.
2. Suppose that Q is a refinement of the partition P of the interval $[a,b]$ and that f is a bounded function on $[a,b]$. Show that
$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$
   Derive, as a corollary that $L(f,P_1) \leq U(f,P_2)$ for *all* partitions P_1 and P_2 of $[a,b]$.
3. Define f to be the step function
$$f(x) = \begin{cases} 1 & \text{if } -3 \leq x < 0 \\ -2 & \text{if } 0 \leq x \leq 1. \end{cases}$$
   Show that f is integrable over $[-3,1]$ and that $\displaystyle\int_{-3}^{1} f = 1$.
4. Suppose that $f_n : [a,b] \to \mathbb{R}$ is integrable on $[a,b]$ for each n and that $f_n \to f$ uniformly on $[a,b]$. Prove that f is integrable on $[a,b]$.
5. Manipulate the geometric series to show that
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}.$$
   Note: You'll need to substitute $x \to -x^2$, integrate the result, and plug in the appropriate number. That number is on the *boundary* of the domain of convergence, though, so you'll need to use Abel's theorem to justify the equality.