Series and elementary function problems

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1. Recall that we showed that $\zeta(z) = \eta(z)/(1 - 2^{1-z})$. I've also stated (without proof) a couple of times that
$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \text{ and } \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$
Use those facts to find closed-form expressions for
$$\eta(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \text{ and } \eta(4) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}.$$

2. Use the definition of the exponential function to show that
   a. $|e^z| \leq e^{|z|}$
   b. $\lim_{z \to 0} \dfrac{e^z - 1}{z} = 1$

3. Compute the values of the following:
   a. $\operatorname{Log}(\sqrt{3} + i)$
   b. $\log(8i)$
   c. $(1 + i)^{\pi i}$
   d. 2^i