Review for Exam 1

2026-09-22

Our first exam is next Wednesday, September 30. Here are some topics and problems to study.

Definitions and theorems

- Abs(z) the absolute value of z
- arg(z) the set of arguments of z
- Arg(z) *the* principal argument of z
- $w^{1/n}$ the principal n^{th} root of w
- Limit the ε-δ definition
- $f'(z_0)$ the derivative of a function at a point
- Function that's analytic at a point z_0 and/or analytic on a region

Problems

1. Find both square roots of $1 + i$. Identify which one is the principal square root.
2. Separate $f(z) = 1 + z + z^3$ into its real and imaginary parts.
3. Writing $f(x + iy) = u(x, y) + iv(x, y)$, where $x, y \in \mathbb{R}$ and u, v map $\mathbb{R}^2 \to \mathbb{R}$, use the Cauchy-Riemann equations to determine which of the following defines an analytic function.
   a. $f(x + iy) = (x^2 - y^2 + e^x \cos(y)) + i\,(2xy + e^x \sin(y))$
   b. $f(x + iy) = (x + y) + i(x - y)$
   c. $f(z) = \bar{z}$
4. Use the definition of the derivative to show that the function $f(z) = \bar{z}$ is nowhere differentiable.

5. Express the following complex numbers in the form $a + bi$.

   a. $1/(1+i)$

   b. $\left(\dfrac{1+i}{\sqrt{2}}\right)^{100}$

6. Let $R = \{z \in \mathbb{C} : 1 < |z| < 2,\ 0 \leq \arg(z) < \pi\}$ and let R^2 denote the image of R under the square function.

   a. Sketch R in the plane. Be sure to indicate any edges *not* contained in R with dashed lines, while edges that *are* contained in R should be solid.

   b. Is R open, closed, or neither? You needn't prove or even justify your assertion.

   c. Sketch R^2 in the plane. Be sure to indicate the image of each edge of R.

   d. Is R^2 open, closed, or neither? You needn't prove or even justify your assertion.

7. Recall that the *cross-ratio* of four complex numbers z, z_1, z_2, z_3 is defined by

$$[z, z_1, z_2, z_3] = \frac{(z - z_1)\,(z_2 - z_3)}{(z - z_3)\,(z_2 - z_1)}.$$

   a. If $T(z) = [z, z_1, z_2, z_3]$, then what are the images of z_1, z_2, and z_3 under T?

   b. Use the cross-ratio to find a Möbius transformation that sends

$$2 \to 0,\ 4 \to 1,\ \text{and } 3 + i \to \infty.$$

   c. What is the image of the circle of radius 1 centered at 3 under the action of your Möbius transformation?

8. Write down an ε-δ proof of the fact that

$$\lim_{z \to i} ((1+i)z + 1) = i.$$

9. Let

$$f(z) = \frac{(1+i)z^2}{2z+1}.$$

   We consider f as a map of the Riemann sphere to itself that fixes ∞.
   Compute $f'(\infty)$ and interpret the result.