Reduction to reduced row echelon form

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Here's a bit of info and examples on Reduced Row Echelon Form (RREF).

The definition

We say that a matrix is in reduced row echelon form if the following properties are satisfied.

- If the entries in a row are all zero, then the same is true of any row below it.
- If we move across a row from left to right, the first nonzero entry we encounter is 1. We call this entry the leading entry in the row.
- The leading entry in any row is to the right of the leading entries in all the rows above it.
- A leading entry is the only nonzero entry in its column.

Some RREF examples??

Which of these look like they are in reduced row echelon form?

$$B_1 = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Performing row reduction

Here area few examples illustrating row reduction to reduced row echelon form.

The typical situation

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - R_1} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & -1 & -3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_3} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - R_3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Infinitely many solutions

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 4R_1,\ R_3 \leftarrow R_3 - 7R_1} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{3} R_2} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -6 & -12 & -6 \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow R_3 + 6R_2} \left[\begin{array}{rrr|r} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 2R_2} \left[\begin{array}{rrr|r} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

An inconsistent system

$$\left[\begin{array}{rrr|r} 1 & 4 & 5 & 7 \\ 2 & 8 & 10 & 14 \\ 3 & 12 & 15 & 22 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1,\ R_3 \leftarrow R_3 - 3R_1} \left[\begin{array}{ccc|c} 1 & 4 & 5 & 7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 4 & 5 & 7 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 7R_2} \left[\begin{array}{ccc|c} 1 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

A randomly chosen system

$$\left[\begin{array}{rrr|r} 8 & 12 & 10 & 7 \\ 12 & 9 & 1 & 10 \\ 5 & 9 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 12 & 9 & 1 & 10 \\ 8 & 12 & 10 & 7 \\ 5 & 9 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - \frac{2}{3}R_1,\ R_3 \leftarrow R_3 - \frac{5}{12}R_1} \left[\begin{array}{rrr|r} 12 & 9 & 1 & 10 \\ 0 & 6 & \frac{28}{3} & \frac{1}{3} \\ 0 & \frac{21}{4} & \frac{7}{12} & -\frac{13}{6} \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{6}R_2} \left[\begin{array}{rrr|r} 12 & 9 & 1 & 10 \\ 0 & 1 & \frac{14}{9} & \frac{1}{18} \\ 0 & \frac{21}{4} & \frac{7}{12} & -\frac{13}{6} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 - 9R_2,\ R_3 \leftarrow R_3 - \frac{21}{4}R_2} \left[\begin{array}{rrr|r} 12 & 0 & -13 & \frac{19}{2} \\ 0 & 1 & \frac{14}{9} & \frac{1}{18} \\ 0 & 0 & -\frac{91}{12} & -\frac{59}{24} \end{array}\right]$$

$$\xrightarrow{R_3 \leftarrow -\frac{12}{91}R_3} \left[\begin{array}{rrr|r} 12 & 0 & -13 & \frac{19}{2} \\ 0 & 1 & \frac{14}{9} & \frac{1}{18} \\ 0 & 0 & 1 & \frac{59}{182} \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - \frac{14}{9}R_3,\ R_1 \leftarrow R_1 + 13R_3} \left[\begin{array}{rrr|r} 12 & 0 & 0 & \frac{96}{7} \\ 0 & 1 & 0 & -\frac{35}{78} \\ 0 & 0 & 1 & \frac{59}{182} \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow \frac{1}{12}R_1} \left[\begin{array}{rrr|r} 1 & 0 & 0 & \frac{8}{7} \\ 0 & 1 & 0 & -\frac{35}{78} \\ 0 & 0 & 1 & \frac{59}{182} \end{array}\right]$$