Review for exam 2

2025-10-22

Our second exam is next Wednesday, October 29! Here's our review sheet.

Mostly computational problems

Most of the problems on the exam will be like one of the following. This section might be a bit longer than what you'll see on the exam.

1. Write down careful and complete definitions of the following:
   a. Eigenvalue/Eigenvector pair for a matrix A Definition 1.4.1
   b. Characteristic polynomial of a matrix A ($\det(A - \lambda I) = 0$)
   c. Similarity of matrices Definition 4.3.11
   d. Diagonalizable matrix Definition 4.3.4
   e. ~~Nonsingular matrix~~
2. Write down three characterizations of *Invertible Matrix*.
3. Suppose
$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 4 \\ 2 & 4 & 1 & 4 & 5 \\ 3 & 6 & 1 & 5 & 6 \end{bmatrix}$$
   and that A has reduced row echelon form
$$\begin{bmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
   a. Find a basis for the null space of A.
   b. Find a basis for the column space of A.
   c. Find a basis for the row space of A.

4. In this problem, you're going to write down a couple of 3×3 matrices that perform specific geometric actions. In both cases, you should write your matrix as a factorization expressing its diagonal form, i.e. as SDS^{-1}.

   a. Write down a matrix that stretches by the factor 3 in the direction $\begin{bmatrix}1, 1, 1\end{bmatrix}^\mathsf{T}$, that stretches by the factor 2 in the direction $\begin{bmatrix}1, 2, 3\end{bmatrix}^\mathsf{T}$, and that reflects in the direction $\begin{bmatrix}3, -2, 1\end{bmatrix}^\mathsf{T}$.

   b. Write down a matrix that stretches by the factor 2 in the direction $\begin{bmatrix}1 & 2 & 3\end{bmatrix}^\mathsf{T}$ and induces pure rotation through the angle $\pi/4$ about the line with direction vector $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\mathsf{T}$.

5. Let
$$A = \begin{bmatrix} -73 & 117 & -168 & 207 \\ 170 & -278 & 400 & -495 \\ 523 & -849 & 1220 & -1510 \\ 301 & -488 & 701 & -868 \end{bmatrix}.$$
Suppose I tell you that A is similar to
$$M = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

   a. What are the eigenvalues of A?
   b. What can you deduce about the geometric action of A?
   c. What information about the geometric action of A can you *not* deduce?

6. Let
$$A = \begin{bmatrix} 1 & -1 \\ 6 & 6 \end{bmatrix}.$$

   a. Diagonalize A.
   b. Evaluate $A^{10} \begin{bmatrix}1 & 1\end{bmatrix}^\mathsf{T}$

7. Use a determinant to write down an equation of the circle containing the points $(1, 1)$, $(1, 2)$, and $(-2, 0)$. You may express your answer using an unexpanded determinant.

More theoretical problems

There will probably be something a bit more theoretical on the exam. These sorts of questions typically require you to write down a definition or two and maybe a theorem and think about what that all tells you. If you're dealing with similarity of matrices, for example, you'll almost certainly need to play with the defining equation
$$A = SBS^{-1}.$$
This section is certainly much longer than what you'll see on the exam.

1. Suppose that A is similar to B and that λ is an eigenvalue of A with corresponding eigenvector $\mathbf{x}$. Show that λ is also an eigenvalue of B. What is the corresponding eigenvector of B?
2. Provide a counterexample to the statement that a square matrix is always similar to its inverse.
3. Suppose that A is a square matrix. Explain why the constant term of the characteristic polynomial of A is equal to the determinant of A.
4. Suppose that A is similar to B and that $n \in \mathbb{N}$. Show that A^n is similar to B^n.
5. Suppose that A is similar to B and that B is similar to C. Show that A is similar to C.