Review for exam 1

2025-09-10

Our first exam is next Wednesday, September 17! Here's our first draft of a review sheet.

Problems

1. State the following definitions:

    a. Pivot position in a matrix. Definition 1.4.1
    b. Linear Combination Definition 2.1.9
    c. The span of a set of vectors Definition 2.3.1
    d. Linear independence of a set of vectors Definition 2.4.5
    e. Homogeneous system of equations Top of section 2.4.3

2. In this problem, we're going to consider the types of solutions that *might* occur and typically *do* occur for linear systems of various sizes.

    a. Suppose that we have a linear system in 3 equations and 5 unknowns.
        i. *Generally*, how many solutions do we expect there to be?
        ii. Is it possible for there to be a *unique* solution?
        iii. Write down a possible RREF of an augmented matrix for such a system that has no solution.
    b. Suppose that we have a linear system in 5 equations and 3 unknowns.
        i. *Generally*, how many solutions do we expect there to be?
        ii. Is it possible for there to be a *unique* solution?
        iii. Write down a possible RREF of an augmented matrix for such a system that has infinitely many solutions.
    c. Suppose that we have a linear system in 4 equations and 4 unknowns.
        i. *Generally*, how many solutions do we expect there to be?
        ii. Write down a possible RREF of an augmented matrix for such a system that has exactly one solution.

3. Write down a componentwise proof of the fact that vector addition is associative. That is, if **u**, **v**, and **w** are vectors in $\mathbb{R}^n$, then

$$(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}).$$

I guess you might try the same thing with commutativity.

4. Consider the vectors that form the columns of the following matrix M:

$$M = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 \\ 2 & -2 & 0 & -2 & 2 \\ -1 & 1 & 0 & 3 & 1 \\ 1 & 3 & 4 & -1 & 1 \end{pmatrix}$$

   a. Without doing a single computation, explain why there's no way for these vectors to be linearly independent.
   b. Now, the RREF of M is shown below. Based on that, find a linearly independent subset of the columns whose span is the same as the span of all the columns.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

5. Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{y} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ and } \mathbf{z} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

   Express the vector

$$\mathbf{u} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

   as a linear combination of **x**, **y**, and **z** or explain why there is no such linear combination.

6. Suppose that A is a matrix of dimensions 5×8 and B is a matrix of size 7×5. Then, what are the dimensions of the matrices

   a. AB and
   b. BA?

7. Let A and B denote the matrices

$$A = \begin{pmatrix} 3 & 0 \\ 3 & -2 \\ 1 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ -3 & -3 \end{pmatrix}.$$

a. Compute AB or explain why that makes no sense.
b. Compute BA or explain why that makes no sense.

8. Write down 2×2 matrices that perform the following actions. In some cases, you might want to express your answer as a product of matrices that perform simpler actions.

   a. Stretches by the factor 2 in the horizontal direction and by the factor 3 in the vertical.
   b. Stretches by the factor 2 in the horizontal direction by the factor 3 in the vertical, and also reflects across the x-axis.
   c. Projects on the line $y = x$.
   d. Reflects across a line through the origin that makes an angle of 17° with the x axis.