The number e and compound interest

Introduction

Investments left in an account that accrues interest at a regular rate grow exponentially. This document explores the effect of "compounding" that interest more regularly has on the effective rate of growth. This turns out to require a good understanding of exponential functions and the natural exponential base e. As such, this topic illustrates a nice application for beginning calculus students.

What is e?

Wikipedia defines e in terms of a limit:

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n.$$

In fact, it is exactly this characterization of e that makes it important in the theory of compound interest. For a beginning calculus student, though, the crucial thing about e is that it is the natural choice as a base for exponential functions. The first thing we'll do here is reconcile those two perspectives.

The natural base of exponential functions

When first considering the derivative of the general exponential function $f(x) = b^x$, where b is a positive base, it's natural to apply the definition of the derivative in terms of the difference quotient:

$$\begin{aligned}\frac{d}{dx}b^x &= \lim_{h\to0}\frac{b^{x+h}-b^x}{h} = \lim_{h\to0}\frac{b^x b^n - b^x}{h} \\ &= b^x \lim_{h\to0}\frac{b^h-1}{h} = c_b\, b^x.\end{aligned}$$

The value of that limit $\lim_{h\to 0}(b^h-1)/h$ is presumably *some* number, which I've chosen to denote with a c_b. Thus, this computation suggests that the derivative of any exponential function is simply a constant times that exponential function you started with.

The limit that this leads to

$$\lim_{h\to 0}\frac{b^h-1}{h}$$

is not quite as elementary as those limits that arise when dealing algebraic functions like polynomials. There's no evident way to cancel the h in the denominator. We might still choose to explore the limit numerically, however. To this end, Table 1 lists values of $(2^h-1)/h$ and $(3^h-1)/h$ near $h=0$.

Table 1: Values of $(2^h-1)/h$ and $(3^h-1)/h$ near $h=0$

h	0.100000	0.010000	0.001000	0.000100	0.000010	0.000001
$(2^h-1)/h$	0.717735	0.695555	0.693387	0.693171	0.693150	0.693147
$(3^h-1)/h$	1.161232	1.104669	1.099216	1.098673	1.098618	1.098613

The table indicates that

$$\lim_{h\to 0}\frac{2^h-1}{h}\approx 0.6931 \text{ and } \lim_{h\to 0}\frac{3^h-1}{h}\approx 1.0986.$$

This suggests that there must be some value of b between 2 and 3 so that the limit

$$\lim_{h\to 0}\frac{b^h-1}{h} \text{ is exactly 1.}$$

We denote this value of b with the letter e. That is, the number e is chosen, by definition, to be the unique number such that

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

The great thing about this from the perspective of calculus is that we have the simpler formula for the derivative of the exponential function, namely

$$\frac{d}{dx}e^x=e^x.$$

Thus $f(x)=e^x$ is the exponential function whose derivative is as simple as possible.

The limit definition

So, how does this approach to e as the natural exponential base relate to Wikipedia's claim that

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n ?$$

Well, if

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1,$$

then for small values of h we must have

$$\frac{e^h - 1}{h} \approx 1.$$

Solving for e, we get

$$e \approx (1 + h)^{1/h}$$

or, maybe

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

The advantage here is that it makes it much easier to compute estimates for the numerical value of e; simply plug values of h that are close to zero into $(1 + h)^{1/h}$. The common definition stated by Wikipedia is simply a sequential version of this obtained with the identification $n = 1/h$. The limit then becomes

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n.$$

We can then approximate e by plugging in large values of n into $(1 + 1/n)^n$.

Table 2: Values of $(1 + 1/n)^n$ for large n

n	10	100	1000	10000	100000	1000000
$(1 + 1/n)^n$	2.593742	2.704814	2.716924	2.718146	2.718268	2.718280

It looks like $e \approx 2.718$. A more precise approximation is

$$e \approx 2.71828182845904523536028747 14.$$

The basics of compound interest

Let's shift gears now to talk about compound interest. The basic question asks, how much money will a bank account hold if we invest a certain amount at a given interest rate and let it grow year by year? To be more precise, suppose that

- P denotes our initial investment (often called the *principal*),
- r denotes the annual interest rate, and
- t denotes the length of time that we leave the account alone.

We'd like a function $A(t)$ that tells us the amount that our account hold as a function of time t. For the time being we'll assume that t takes discrete steps in numbers of years.

Analysis

Let's be clear about the annual interest rate. This is simply the factor by which the money grows from one year to the next.

- At time $t = 0$, we've just put the money in the bank. We have $A(0) = P$.
- After $t = 1$ year, $A(1) = P + rP = P(1 + r)$.
- After $t = 2$ years, $A(2) = P(1 + r) + rP(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$.
- $\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$
- After t years, $A(t) = P(1 + r)^t$.

Note that the interest that we make each year rolls into the investment the next year. This is the defining characteristic of *compound* interest. The effect is that your investment grows exponentially! Generally, though, the base $1 + r$ is just a little larger than 1.

Example: Suppose we invest \$1000 at an annual interest rate of 4%. How much will we have after

- 1 year?
- 10 years?

Solution:

- After 1 year, we have $A(1) = 1000 \times 1.04 = 1040$.
- After 10 years, we have $A(10) = 1000 \times (1.04)^{10} = 1480.24$.

Frequency of compound interest

In the previous example, the interest compounded *annually*. What if we want more frequent access to our funds, though? The typical approach is to compound the funds more frequently; we divide our rate, though, by the number of times we compound annually.

Let's suppose that our annual rate is again 4%. Now, though, we wish to compound the interest *quarterly*, that is 4 times throughout the year. The standard way to accomplish this is to use a quarterly rate of 1%. Then, after one year, the interest will have compounded once. After ten years, the interest will have compounded 40 times. We'll find that

- After 1 year, we have $A(1) = 1000 \times (1.01)^4 = 1040.60$.
- After 10 years, we have $A(10) = 1000 \times (1.04)^{40} = 1488.86$.

Not surprisingly, we make a bit more money as time goes on. The change is not huge, though; the main advantage is more frequent access.

Of course, we might want even more frequent access. We might want to compound monthly, or daily, or minutely, or anything really. We really should write down a general formula in terms of a parameter n representing the number of times per year that we wish to compound the interest. This is not hard:

If we compound the interest n times per year at an annual rate r, then

- the rate per period is r/n,
- the number of times we compound the interest per year is nt, so
- $$A(t) = P(1 + r/n)^{nt}. \tag{1}$$

Example: Suppose that on Jan 1 we invest \$500 at an annual interest rate of 6% compounded daily. How much will we have on July 5 of a non leap year?

Solution: Our rate per period will be 0.06/365 and July 5 is the 185$^{\text{th}}$ day of the year or 184 days after Jan 1. Thus we'll have

$$500 \times (1 + 0.06/365)^{184} = \$515.35.$$

Example: Suppose that we invest \$1 on Jan 1 at an annual interest rate of 5% compounded 12 times per year or approximately monthly. How much will we have on Jan 1 of the next year?

Solution: Our rate per period will be 0.05/12 and we'll compound 12 times. Thus we'll have

$$(1 + 0.05/12)^{12} = \$1.05116.$$

Again, we get a little more than the \$5 that we'd get if we'd compounded just once. In fact, the *effect* is as if our interest were 5.116%. This is called the *effective rate of interest*.

Interest compounded *continuously*

What if we want *constant* access to our money? Compounding by the minute is insufficient; we want our money *now*!

The solution is to take the limit as $n \to \infty$ in Equation 1. Here's that computation:

$$\begin{aligned} \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} &= \lim_{n\to\infty} P\left(1 + \frac{1}{n/r}\right)^{\frac{n}{r}rt} \\ &= \lim_{n\to\infty} P\left(\left(1 + \frac{1}{n/r}\right)^{n/r}\right)^{rt} = Pe^{rt}. \end{aligned}$$

What a beautifully simple and pert little formula! To get there we used the fact that

$$\left(1 + \frac{1}{n/r}\right)^{n/r} \to e \text{ as } n \to \infty.$$

That's just the definition of e, since $\frac{n}{r} \to \infty$ as $n \to \infty$.

Example: Suppose that \$1000 was automatically deposited in a new account as the clock struck midnight on New Year's Eve going into 2009. The account accrues interest at an annual rate of 6.5% compounded continuously. How much would the account hold at 9:17 AM on July 5?

Solution:

Of course, you can do this on a calculator. This example is a bit more complicated, though, so here's how I'd approach it with Python:

```python
from numpy import exp

# Minutes in a year
M = 24*60*365

# Minutes until 9:17 AM on July 5
m = 24*60*184 + 9*60 + 17

# Time in years
t = m/M

# Annual rate
r = 0.065

# Amount after time t if compounded continuously:
print(1000*exp(r*t))
```

```
1033.3810575360308
```