Chaos and Fractals - Review 2

1. Write down a careful definition of each of the following
   (a) Iterated function system
   (b) Similarity dimension of an IFS with contraction ratios $\{r_1, r_2, \ldots, r_m\}$
   (c) Box counting dimension
   (d) The strong open set condition
   (e) The dimension comparison theorem

2. Find the value of r so that the IFS $\{rx, x/2 + 1/2\}$ yields an invariant set with similarity dimension 1/2.

3. Consider the following IFS of four Affine functions:

$$f_1(\vec{x}) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \vec{x}$$
$$f_2(\vec{x}) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$$
$$f_3(\vec{x}) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$
$$f_4(\vec{x}) = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \vec{x} + \begin{pmatrix} \frac{3}{4} \\ \frac{1}{2} \end{pmatrix}$$

   Several iterations of this IFS applied to the unit square are shown in figure 1.

   (a) Show that n^{th} cover of E generated by this process consists of 4^n rectangles of width $1/(4n)$ and height $1/(2n)$.
   (b) How many squares of side length $1/(4n)$ must each of the rectangles in the previous part be decomposed into to cover the attractor with squares?
   (c) Use the squares from the previous part to compute $N_{4^{-k}}(E)$. (Don't count squares that intersect E only in their boundaries.)
   (d) Compute the box counting dimension of E.

4. Suppose we generate a Digraph IFS by associating some similarity transformation with common contraction ratio r for each of the edges in the directed graph G shown in figure 2. Write down the incidence matrix associated with G and use it to find the common digraph-similarity dimension of the digraph fractals generated by this Digraph IFS.

   Note: You can use this Python code to check your eigenvalue finding skills: `https://sagecell.sagemath.org/?q=ybjrla`.

5. The self-similar set shown in figure 3 is called the Koch snowflake. It consists of seven copies of itself, six of which are scaled by the factor $1/3$ and one of which is scaled by the factor $1/\sqrt{3}$.

   (a) Write down a formula that the dimension of the Koch Snowflake must satisfy.

   (b) What do you think the dimension of the Koch Snoflake should be?

   (c) Use your formula to show that your dimension conjecture is true.

   Note: The boundary of the Koch Snowflake consists of Koch curves but it's the solid plane region shown in the figure that we are studying here.

6. The effect of an IFS of four similarity transformations, each with contraction ratio $1/2$, is shown on the left in figure 4; the attractor of the IFS is shown on the right.

   (a) Write down a the IFS as a list of functions using mathematical notation involving matrices and vectors.

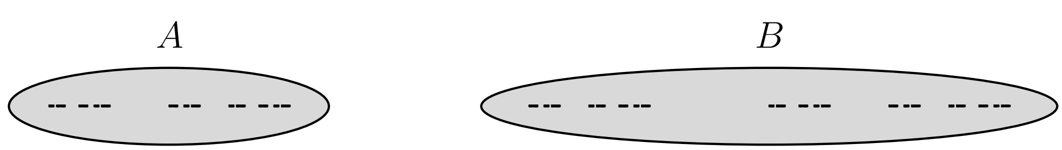
   (b) What is the similarity dimension of the IFS? (I recommend that you simplify the answer.)

   (c) Explain why the box-counting dimension of the attractor might or might not agree with your answer to part (a). Be sure to specifically address:

      - What conditions must be satisfied to ensure equality of similarity dimension and box-counting dimension and
      - Whether the IFS illustrated in the figure satisfies that condition.

7. Consider the IFS on $\mathbb{R}$ given by $f_1(x) = x/4$ and $f_2(x) = x/2 + 1/2$. The invariant set of this IFS consists of two pieces - one scaled by the factor $1/4$ and the other scaled by the factor $1/2$. An approximation to this set looks like so:

   We've computed the fractal dimension of this thing a couple of times before - using box-counting dimension and using similarity dimension. This time, we'll use digraph self-similarity dimension. We can do so by treating the two parts of the set as two separate sets, A and B:



   (a) Write down a digraph IFS with two nodes that has the sets A and B as its attractors.

   (b) Write down the incidence matrix of the digraph IFS.

   (c) Compute the common fractal dimension of the attractors.

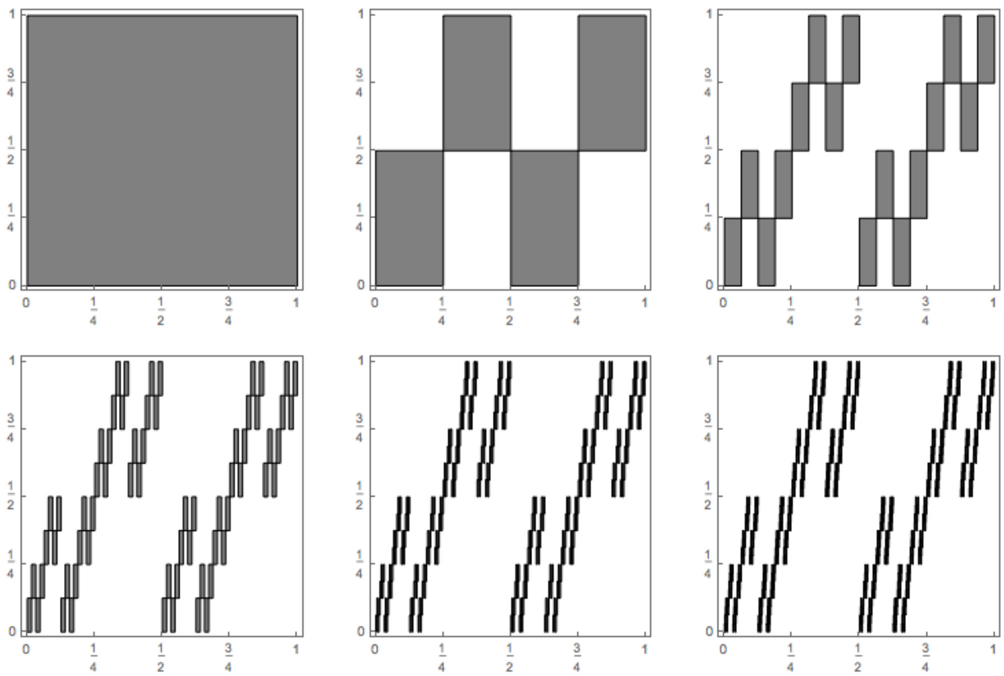


Figure 1: Approximations to a self-affine set

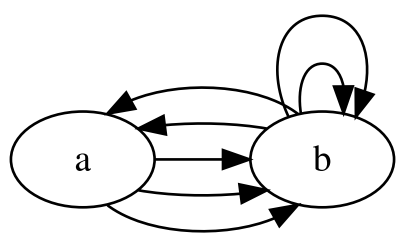


Figure 2: A directed graph

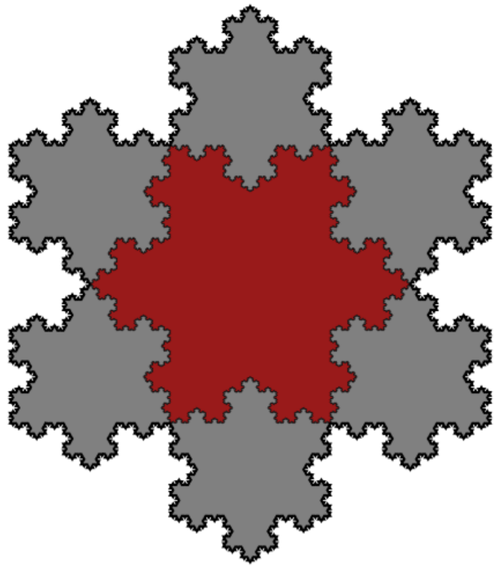


Figure 3: The Koch snowflake

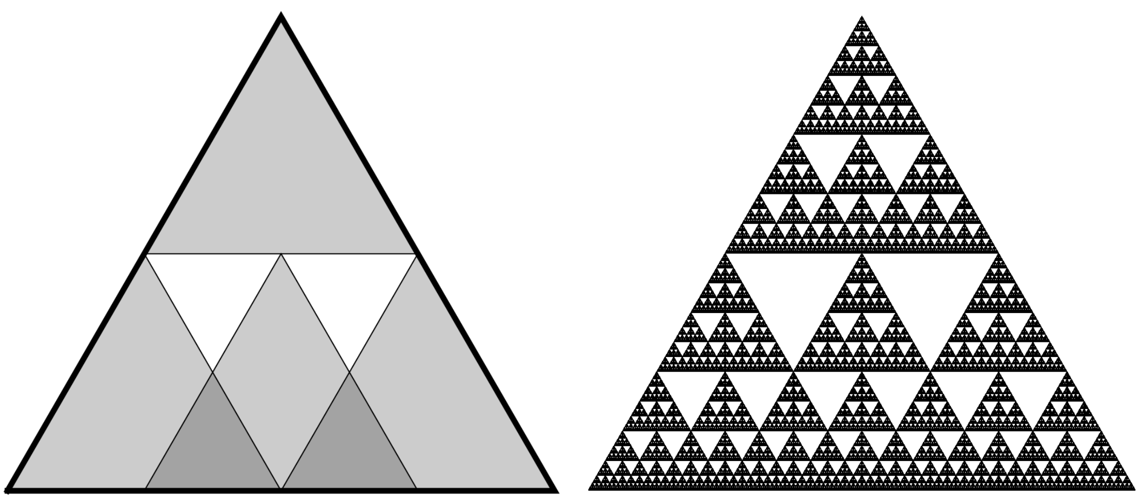


Figure 4: The effect of an IFS on an equilateral triangle and its ultimate attractor