

# Chaos and Fractals

# Chaos and Fractals

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# Preface

This is an introductory text on the basics of chaotic dynamics and fractal geometry. The intended audience includes undergraduate students of mathematics or other technical disciplines with a strong background in calculus and some exposure (or willingness to explore) more advanced topics such as the basics of real analysis, the complex plane, and linear algebra.

Topics in the text include real iterative dynamics (experimentation, cobweb plots, bifurcation, and chaos), complex iterative dynamics (Julia sets, the Mandelbrot set, and the iteration of higher order polynomials), fractal geometry (self-similarity, iterated function systems and fractal dimension).

Another important aspect of this text is its emphasis on computation. We introduce Python code that runs live in the online version to generate images of many of the sets that we'll meet. Such code is generally quite simple using basic constructs such as function definition, conditionals, and iterative loops.

This text was written with [PreTeXt](#) which makes groovy things like live Python code and nicely formatted online and print versions easy.

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# Chapter 1

## Introduction

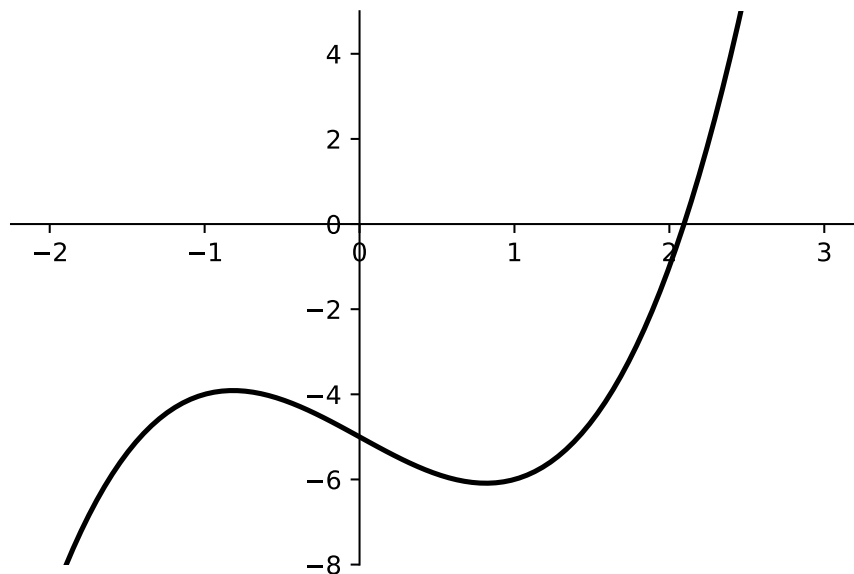
One of the central ideas in both chaos theory and fractal geometry is iteration - i.e., the repetition of a process over and over again. It might seem a bit odd to base so much of our discussion on a process that doesn't necessarily appear to be of fundamental importance to the uninitiated. Thus, in this introduction, we start with a motivation for iteration - namely, Newton's method for solving equations. From that natural beginning, we'll meet many of the important ideas that we'll study further in text.

### 1.1 Surprise in Newton's method

Newton's method is a technique to estimate roots of a differentiable function. Invented by Newton in 1669, it remains a stalwart tool in numerical analysis. When used as intended it's remarkably efficient and stable. After a little experimentation, Newton's method yields some surprises that are very nice illustrations of chaos.

#### 1.1.1 The basics of Newton's method

Let's begin with a look at the example that Newton himself used to illustrate his method. Let  $f(x) = x^3 - 2x - 5$ . The graph of  $f$  shown in [Figure 1.1](#) seems to indicate that  $f$  has a root just to the right of  $x = 2$ .



**Figure 1.1** Newton's example function for his method

Newton figured that, if the root of  $f$  is a little bigger than 2, then he could write it as  $2 + \Delta x$ . He then plugged this into  $f$  to get

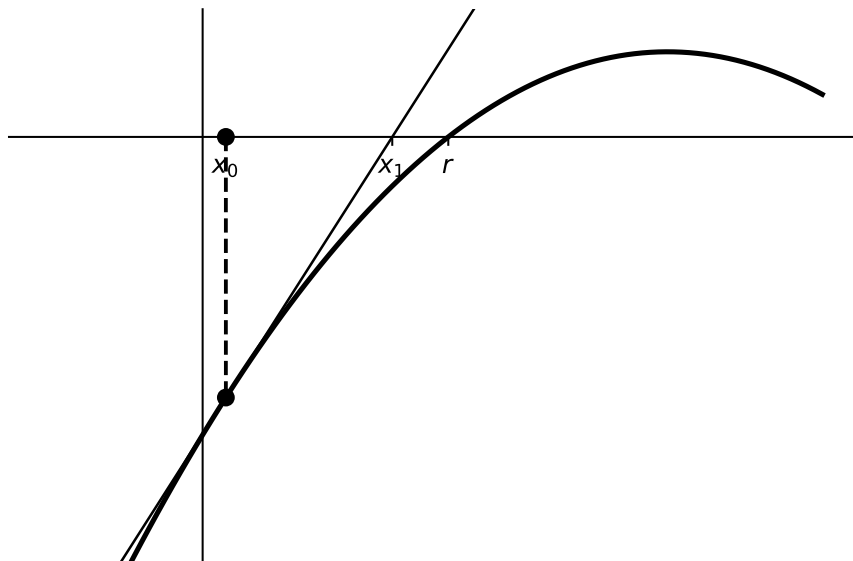
$$\begin{aligned} f(2 + \Delta x) &= (2 + \Delta x)^3 - 2(2 + \Delta x) - 5 \\ &= 8 + 3 \times 2^2 \Delta x + 3 \times 2 \Delta x^2 + \Delta x^3 - 4 - 2\Delta x - 5 \\ &= -1 + 10\Delta x + 6\Delta x^2 + \Delta x^3 \\ &\approx -1 + 10\Delta x. \end{aligned}$$

In that last step, since  $\Delta x$  is small, he figures that higher powers of  $\Delta x$  are negligibly small. Thus, he figures that  $-1 + 10\Delta x \approx 0$  so that  $\Delta x \approx 1/10$ .

The point is that, if 2 is a good guess at a root of  $f$ , then  $2 + 1/10 = 2.1$  should be an even better guess. A glance at the graph seems to verify this. Of course, we could then repeat the process using 2.1 as the initial guess. We should get an *even better* estimate. The process can be repeated as many times as desired. This is the basis of *iteration*.

### 1.1.2 Newton's method in the real domain

Let's take a more general look at Newton's method. The problem is to estimate a root of a real, differentiable function  $f$ , i.e. a value of  $x$  such that  $f(x) = 0$ . Suppose also that we have an initial guess  $x_0$  for the actual value of the root, which we denote  $r$ . Often, the value of  $x_0$  will be based on a graph. Since  $f$  is differentiable, we can estimate  $r$  with the root of the tangent line approximation to  $f$  at  $x_0$ ; let's call this point  $x_1$ . When  $x_0$  is close to  $r$ , we often find that  $x_1$  is even closer. This process is illustrated in [Figure 1.2](#) where it indeed appears that  $x_1$  is much closer to  $r$  than  $x_0$ .



**Figure 1.2** One step in Newton's method

We now find a formula for  $x_1$  in terms of the given information. Recall that  $x_1$  is the root of the tangent line approximation to  $f$  at  $x_0$ ; let's call this tangent line approximation  $\ell$ . Thus,

$$\ell(x) = f(x_0) + f'(x_0)(x - x_0)$$

and  $\ell(x_1) = 0$ . Thus, we must simply solve

$$f(x_0) + f'(x_0)(x - x_0) = 0$$

for  $x$  to get  $x_1 = x_0 - f(x_0)/f'(x_0)$ .

Now, of course,  $x_1$  is again a point that is close  $r$ . Thus, we can repeat the process with  $x_1$  as the guess. The new, better estimate will then be

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

The process can then repeat. Thus, we can define a sequence  $(x_n)$  *recursively* by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This process is called *iteration* and the sequence it generates often converges to the root  $r$ .

### Examples

We now present several examples to illustrate the variety of things that can happen when we apply Newton's method. Throughout, we have a function  $f$  and we defined the corresponding Newton's method iteration function

$$N(x) = x - \frac{f(x)}{f'(x)}.$$

We then iterate the function  $N$  from some starting point or, perhaps, from several starting points.



**Example 1.3** We start with  $f(x) = x^2 - 2$ . Of course,  $f$  has two roots, namely  $\pm\sqrt{2}$ . Thus, we might think, of the application of Newton's method to  $f$  as a tool to find good approximations to  $\sqrt{2}$ .

First, we compute  $N$ :

$$\begin{aligned} N(x) &= x - f(x)/f'(x) = x - \frac{x^2 - 2}{2x} \\ &= x - \left(\frac{x^2}{2x} - \frac{2}{2x}\right) = x - \left(\frac{x}{2} - \frac{1}{x}\right) = \frac{x}{2} + \frac{1}{x}. \end{aligned}$$

Now, suppose that  $x_0 = 1$ . Then,

$$\begin{aligned} x_1 &= N(1) = \frac{1}{2} + \frac{1}{1} = \frac{3}{2} \\ x_2 &= N(3/2) = \frac{3/2}{2} + \frac{1}{3/2} = \frac{17}{12} \\ x_3 &= N(17/12) = \frac{17/12}{2} + \frac{1}{17/12} = \frac{577}{408} \end{aligned}$$

Note that

$$(577/408)^2 = 332929/166464 = 2 + 1/166464$$

so that third iterate is quite close to  $\sqrt{2}$ .

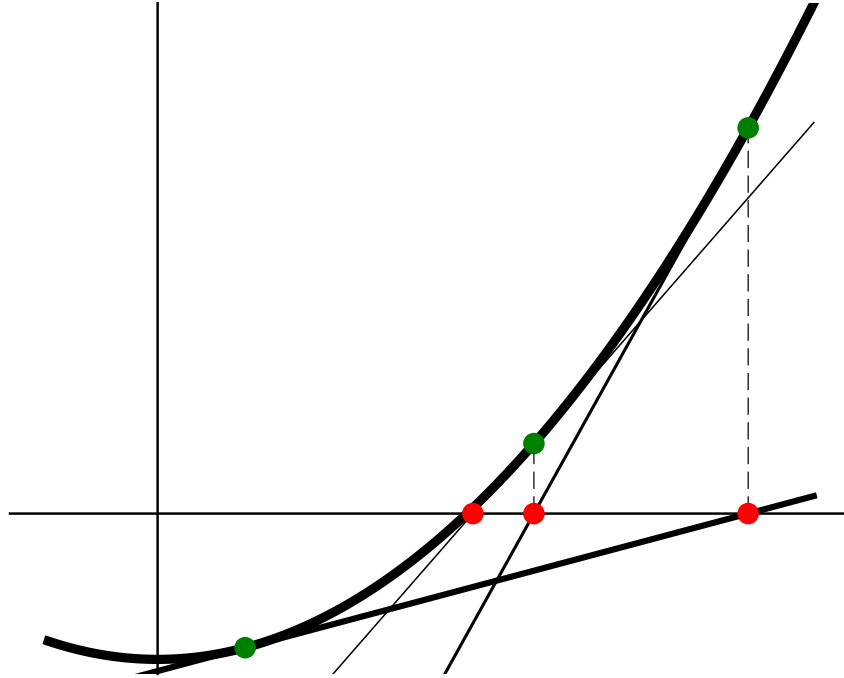
Note that we've obtained a *rational* approximation to  $\sqrt{2}$ . At the same time, it's clear that it would be nice to perform these computations on a computer. In that context, we might generate a *decimal* approximation to  $\sqrt{2}$ . Here's how this process might go in Sage:

```
f(x) = x^2 - 2
N(x) = x - f(x)/diff(f(x), x)
xi = 2.0
for i in range(7):
    xi = N(xi)
    print(xi)
```

```
1.5000000000000000
1.4166666666666667
1.41421568627451
1.41421356237469
1.41421356237310
1.41421356237309
1.41421356237310
```

Note how quickly the process has converged to 12 digits of precision.

Of course,  $f$  has two roots. How can we choose  $x_0$  so that the process converges to  $-\sqrt{2}$ ? You'll explore this question computationally in [Exercise 1.3.1](#). It's worth noting, though, that a little geometric understanding can go a long way. [Figure 1.4](#), for example, shows us that if we start with a number  $x_0$  between zero and  $\sqrt{2}$ , then  $x_1$  will be larger than  $\sqrt{2}$ . The same picture shows us that any number larger than  $\sqrt{2}$  leads to a sequence that converges to  $\sqrt{2}$ .



**Figure 1.4** Three steps in Newton's method for  $f(x) = x^2 - 2$

□

**Example 1.5** We now take a look at  $f(x) = x^2 + 3$ . A simple look at the graph of  $f$  shows that it doesn't even hit the  $x$ -axis; thus,  $f$  has no roots. It's not at all clear what to expect from Newton's method.

A simple computation shows that the Newton's method iteration function is

$$N(x) = \frac{x}{2} - \frac{3}{2x}.$$

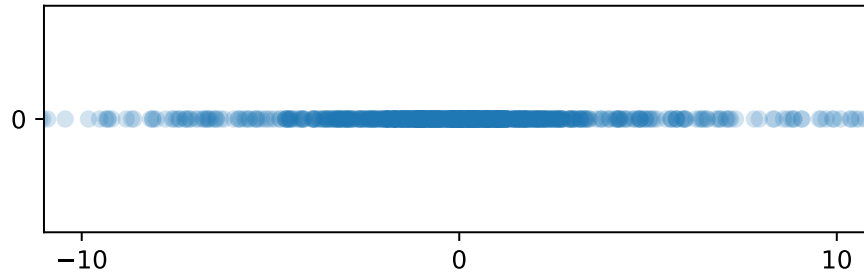
Note that  $N(1) = -1$  and  $N(-1) = 1$ . In the general context of iteration that we'll consider later, we'll say that the points 1 and  $-1$  lie on an orbit of period 2 under iteration of the function  $N$ .

Sequences of other points seem more complicated so we turn to the computer. Suppose that we change the initial seed  $x_0 = 1$  just a little tiny bit and iterate with Sage.

```
f(x) = x^2 + 3
N(x) = x - f(x)/diff(f(x),x)
xi = 2.0
for i in range(100):
    xi = N(xi)
    print(xi)
```

Long crazy **list** of random-looking numbers

Well, there appears to be no particular pattern in the numbers. In fact, if we generate 1000 iterates and plot those that lie within 10 units of the origin on a number line, we get [Figure 1.6](#). This is our first illustration of chaotic behavior. Not just because we see points spread all throughout the interval but also because we appeared to have a stable orbit when  $x_0 = 1$ . Why should the behavior be so different when we change that initial seed to  $x_0 = 0.99$ ?



**Figure 1.6** Chaotic behavior from Newton's method

□

**Example 1.7** Newton's original example had one real root, [Example 1.3](#) had two real roots and [Example 1.5](#) had no real roots. Let's take a look at an example with *lots* real roots, namely  $f(x) = \cos(x)$ .

Generally, the closer your initial seed is to a root, the more likely the sequence starting from that seed is to converge to that root. What happens, though, if we start some place that's not so close to a root? What if we start close to the maximum - near zero? Let's investigate in code.

```
seed(1)
N(x) = x + cot(x)
for j in range(10):
    xi = random()/10
    for i in range(8):
        xi = N(xi)
    print([xi, cos(xi)])
```

```
23.5619449019235, 8.57871740039736e-16]
[14.1371669411541, 5.51091059616309e-16]
[87432.0943457307, -3.22999223447169e-12]
[32.9867228626928, -4.90477700295530e-16]
[67.5442420521806, -4.40934745756706e-15]
[108.384946548848, 2.44867461765812e-15]
[58.1194640914112, 4.89239739022353e-16]
[29.8451302091030, 6.12942380210265e-16]
[36.1283155162826, -3.18470065841971e-15]
[14.1371669411541, 5.51091059616309e-16]
```

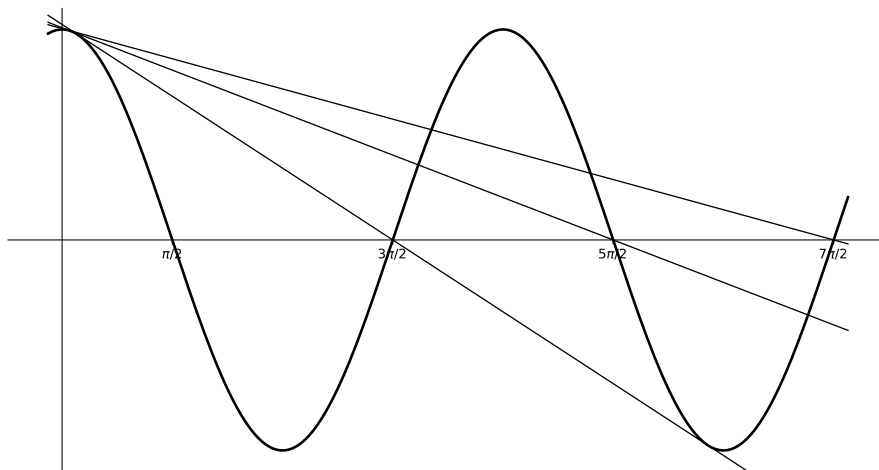
OK, let's pick this code apart. The inner for loop looks like so:

```
xi = random()/10
for i in range(8):
    xi = N(xi)
```

Thus,  $xi$  is set to be a random number between 0 and 0.1; the for loop then iterates the Newton's method function in for the cosine from that initial seed 8 times. The outer for loop simply performs this experiment 10 times. After each run of the experiment, we print the resulting  $xi$  - along with the value of the cosine at that point, to check that we're indeed close to a root of the cosine. It's striking that we get 9 different results over 10 runs even though the starting points are so close to one another.

[Figure 1.8](#) gives some clue as to what's going on. Recall that we can envision a Newton step for a function  $f$  from a point  $x_i$  by drawing the line that is tangent to the graph of  $f$  at the point  $(x_i, f(x_i))$ . The value of  $x_i$  is then the point of intersection of this line with the  $x$ -axis. Because the slope at the

maximum is zero, the value of this point of intersection is very sensitive to small changes. In fact, there are infinitely many roots of the cosine any one of which could be hit by some initial seed in this tiny interval.



**Figure 1.8** Initial Newton steps for the cosine

□

### 1.1.3 Newton's method in the complex domain

Let's move now to the complex domain, where even crazier things can happen. To understand this stuff, of course, you'll need a basic understanding of complex numbers but it's really not a daunting amount of information. You'll need to know that a complex number  $z$  has the form

$$z = x + iy,$$

where  $x$  and  $y$  are real numbers (the real and imaginary parts of  $z$ ) and  $i$  is the imaginary unit (thus  $i^2 = -1$ ). You'll also need to know (or accept) that you can do arithmetic with complex numbers and plot them in a plane, called the complex plane. We'll discuss complex variables in a more detail when we jump into complex dynamics more completely.

#### Cayley's question

In the 1870s, the prolific British mathematician Arthur Cayley (whose name is all over abstract algebra) posed the following interesting question: Given a Suppose that  $p$  is a complex polynomial and  $z_0$  is an initial seed that we might use for Newton's method. Generally,  $p$  will have several roots scattered throughout the complex plane. Is it possible to tell to which of those roots (if any) Newton's method will converge to when we start at  $z_0$ ?

Generally, the closer an initial seed is to a root, the more likely that seed will lead to a sequence that converges to the root. It might be reasonable to guess that the process always converges to the root that is closest to the initial seed. Cayley proved that this is true for quadratic functions but was commented that the general question is much more difficult for cubics. With the power of the computer, there is a very colorful way to explore the question

#### Algorithm 1.9 Coloring basins of attraction for Newton's method.

1. Given: A function  $f : \mathbb{C} \rightarrow \mathbb{C}$ .

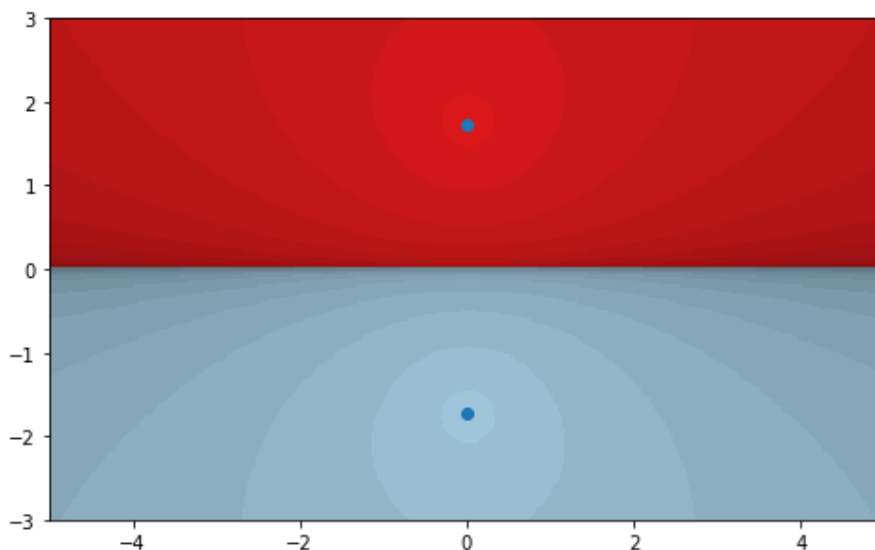
2. Choose a rectangle  $R$  in the complex plane with sides parallel to the real and imaginary axes.
  - The points in  $R$  represent potential initial seeds when applying Newton's method to  $f$ .
3. Discretize the rectangle into points of the form

$$z_{0,jk} = (a + j\Delta x) + (b + k\Delta y)k.$$

4. For each point  $z_{0,jk}$  perform Newton's iteration to generate a sequence  $(z_{n,jk})_n$  until one of two things happen:
  - $|f(z_{n,jk})| < \varepsilon$ , where  $\varepsilon$  is some pre-specified small number. We assume that the process has converged to a root; color the initial seed  $z_{0,jk}$  according to which root the process converged to and shade the initial seed according to how many iterates this process took.
  - The iteration count exceeds some pre-specified bailout; we color the initial seed  $z_{0,jk}$  black.

Let's take a look at a simple, quadratic example.

**Example 1.10** Let  $p(z) = z^2 + 3$ . We already played with this function in the real case back in [Example 1.5](#) and the computer indicated that there might be chaos on the real line. When we allow complex inputs, though, there are two roots - one at  $\sqrt{3}i$  (above the real axis) and one at  $-\sqrt{3}i$  (below the real axis). If the process always converges to the root that is closest to the initial seed, then seeds in the upper half plane should converge to  $\sqrt{3}i$  while seeds in the lower half plane should converge to  $-\sqrt{3}i$ . We don't have the tools to prove this yet, but we can investigate with [Algorithm 1.9](#). The result is shown in [Figure 1.11](#).



**Figure 1.11** Attractive basins for  $f(z) = z^2 + 3$

□

Now, let's take a look at a classic, cubic example.

**Example 1.12** Let  $p(z) = z^3 - 1$ . As a complex, cubic polynomial, we expect  $p$  to have three roots. It's easy to see that  $z = 1$  is a root. This means that

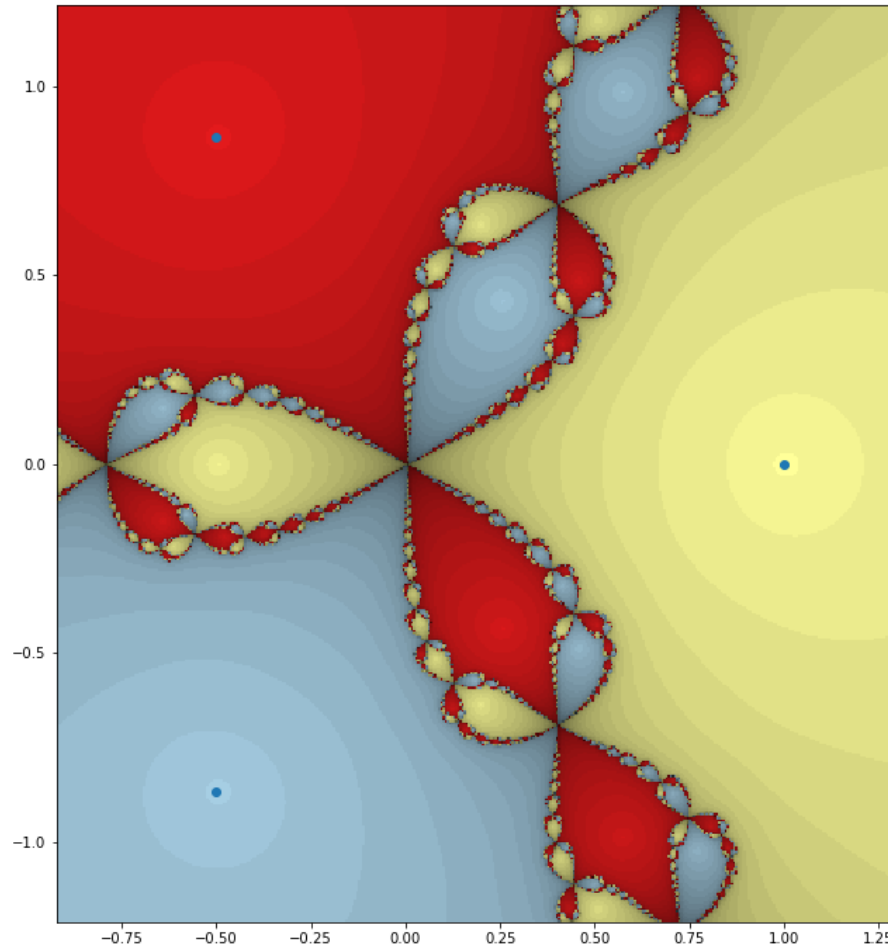
$z - 1$  is a factor, which makes it easier to find that

$$p(z) = (z - 1)(z^2 + z + 1).$$

We can then apply the quadratic formula to find that the other two roots are

$$\frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i.$$

Note that all three roots can be clearly seen in [Figure 1.13](#), which shows the result of [Algorithm 1.9](#).



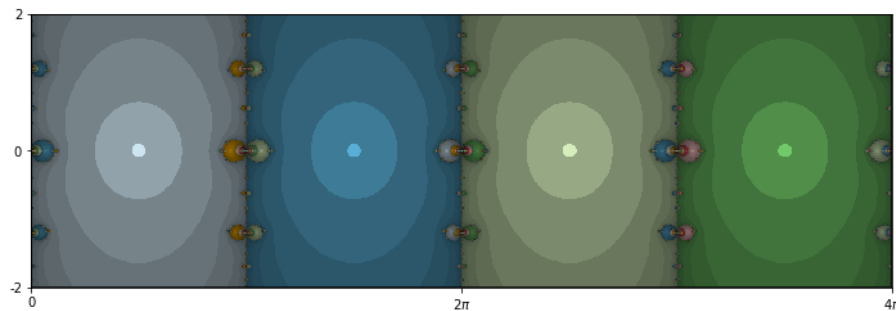
**Figure 1.13** Attractive basins for  $f(z) = z^2 + 3$

□

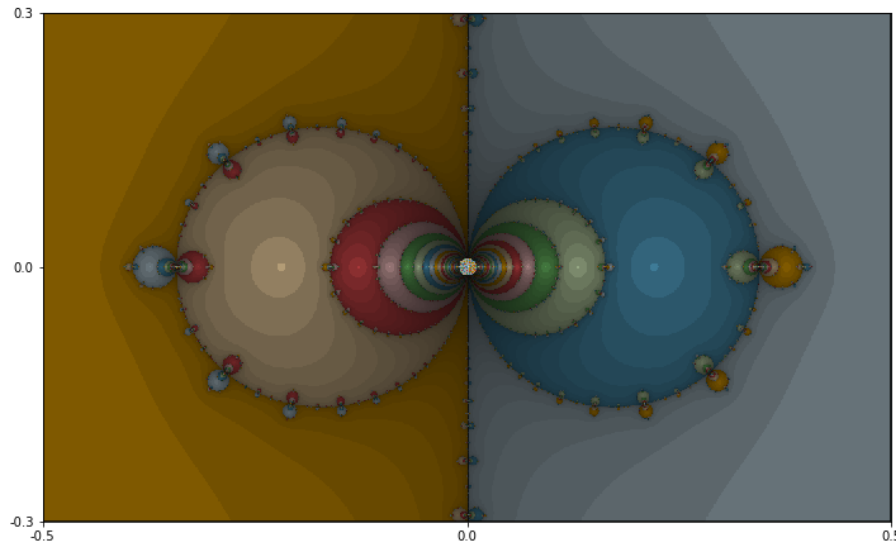
[Figure 1.13](#) is just an incredible picture. We can clearly see the three roots of  $p$  and, indeed, if we start close enough to a root we do converge to that root. The boundary between the basins seems to be incredibly complicated, though.

As a final example, let's take a look at the complex cosine.

**Example 1.14** Let  $f(z) = \cos(z)$ . Of course, we played with the real cosine back in [Example 1.7](#). There, we saw that in any tiny interval containing zero, an initial seed might converge to any of infinitely many roots. In the complex case, the basins of attraction are shown in [Figure 1.15](#), and a zoom is shown in [Figure 1.16](#).



**Figure 1.15** Attractive basins for the cosine



**Figure 1.16** A zoom into the basins of the cosine

Two more incredible pictures! Note the periodicity in [Figure 1.15](#). Each root seems to have its own column of attraction. The boundaries between the basins are again very complicated. In the zoom of [Figure 1.16](#), it looks like there's an infinite sequence of circles collapsing on either side of the origin. This agrees with our observations in the real case of [Example 1.7](#).  $\square$

Generating pictures for the Newton's method is great fun. There is an online tool that allows you to generate the basins of attraction for an (almost) arbitrary polynomial here: [https://marksmath.org/visualization/complex\\_newton/](https://marksmath.org/visualization/complex_newton/).

## 1.2 The scope of chaos

The previous section introduced you a few fascinating ideas and images (functional iteration in real and complex variables and infinitely convoluted shapes that have become known as fractal) - all in the context of Newton's method. The purpose of this short section is to indicate and illustrate a few other areas in which chaos and fractals occur.

### 1.2.1 The iteration of real functions

We were introduced to the idea of real iteration back in [Subsection 1.1.2](#) but there's no reason the function we iterate needs to come from Newton's method.

Figure 1.17 illustrates iteration using a geometric tool called a *cobweb plot*. In part (a) we see the iteration of  $N(x) = x/2 + 1/x$ , that arises in Newton's method. In part (b) we see the iteration of  $f(x) = 3.1x(1 - x)$ . This is an example of a *logistic* function and really has nothing directly to do with Newton's method.

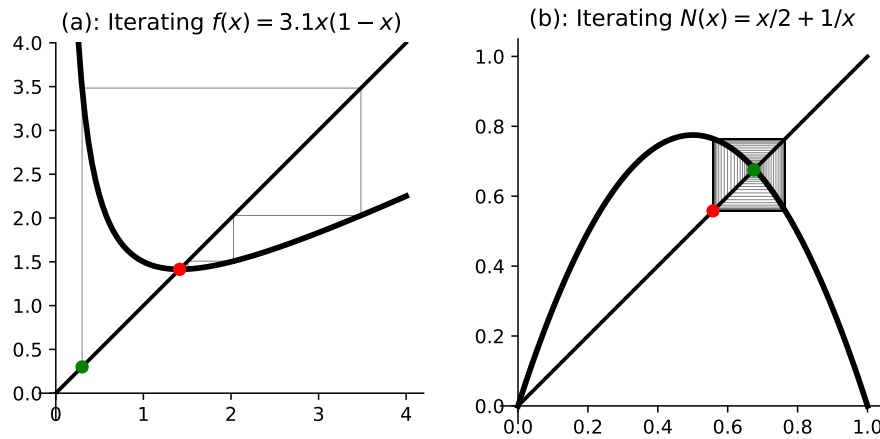


Figure 1.17 Two cobwebs that arise in real iteration

The logistic function above is a member of a *family* of functions - the logistic family. These functions have the form  $f_r(x) = rx(1 - x)$ . When studying a family of functions like this we are often interested in the various types of behavior that might arise. We can explore these possibilities with a tool called a *bifurcation diagram*. The bifurcation diagram for the logistic family is shown in Figure 1.18.

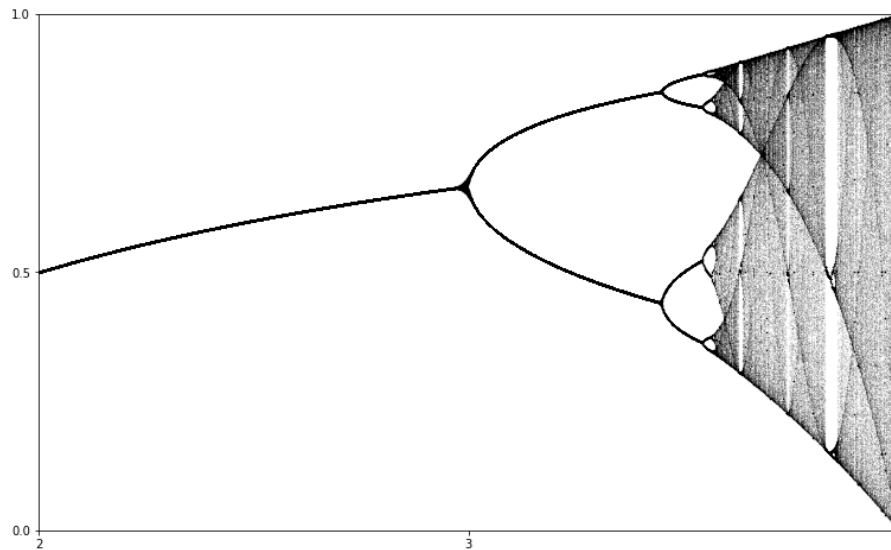


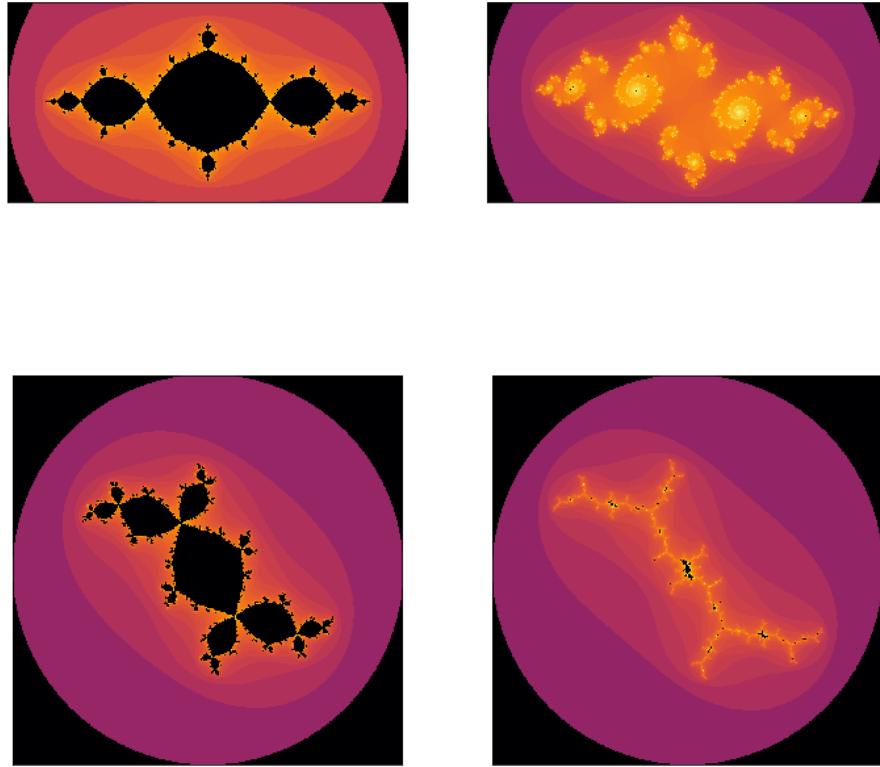
Figure 1.18 The bifurcation diagram for the logistic family

### 1.2.2 The iteration of complex functions

We were introduced to the idea of complex iteration back in Subsection 1.1.3 and, again, there's no reason the function we iterate needs to come from Newton's method. In the context of complex iteration we'll meet Julia sets, a few of

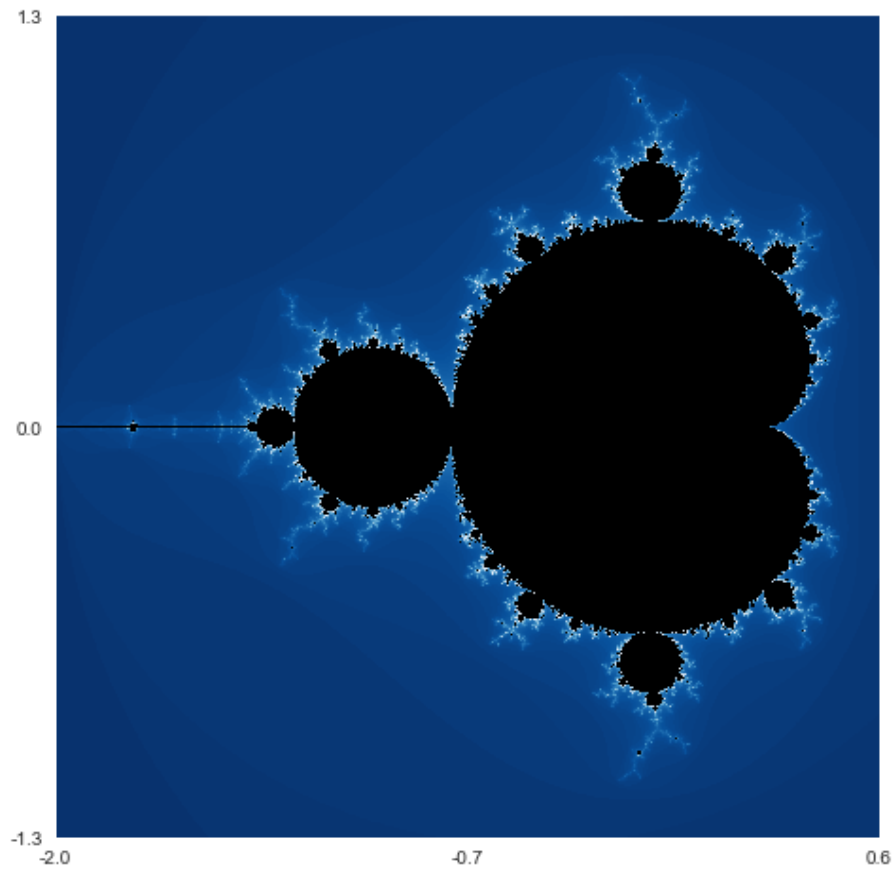


which are shown in [Figure 1.19](#)



**Figure 1.19** Some Julia sets

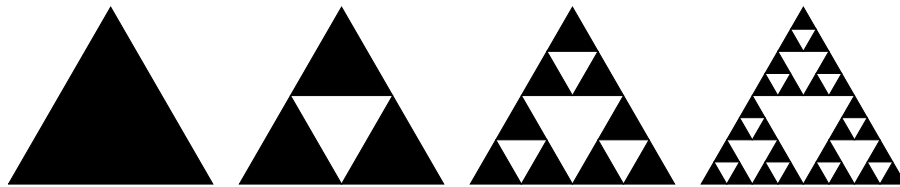
The Julia sets shown in [Figure 1.19](#) all arise from the iteration of functions in the quadratic family - i.e. from the iteration of functions of the form  $f(z) = z^2 + c$ . It is in this context that the famous Mandelbrot set shown in [Figure 1.20](#) arises.



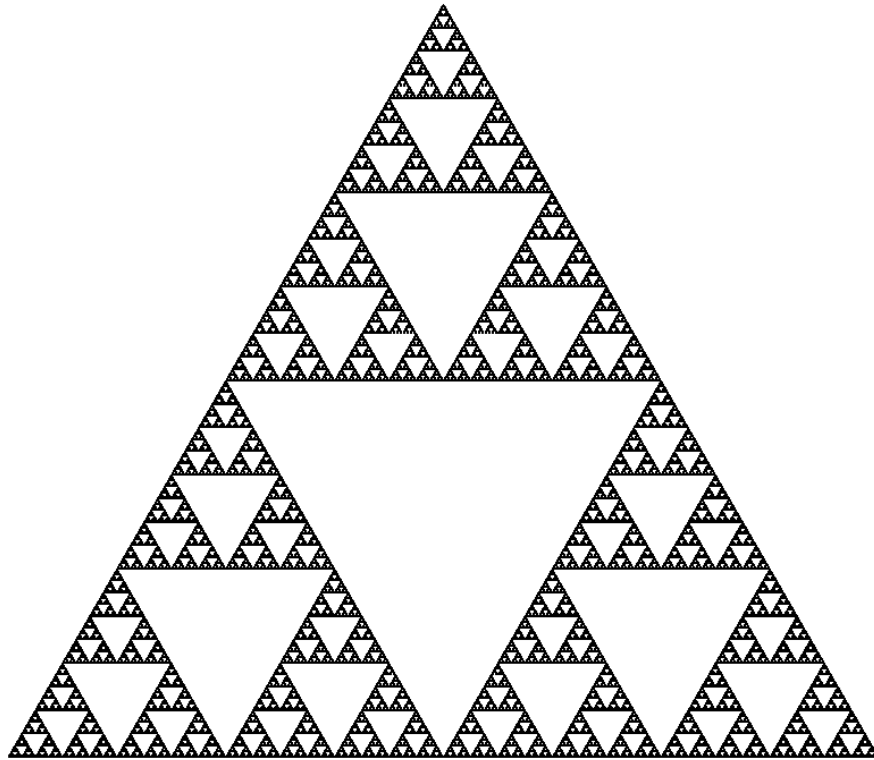
**Figure 1.20** The Mandelbrot set

### 1.2.3 Geometric iteration and fractal geometry

When studying fractal geometry, we might iterate a function that maps sets to other sets. Such an example is shown in [Figure 1.21](#). The limiting figure, called the Sierpinski triangle is shown in [Figure 1.22](#).

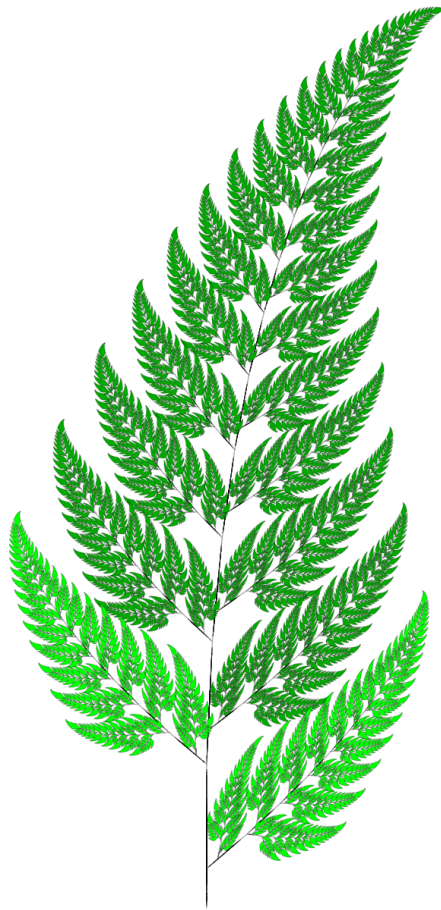


**Figure 1.21** Geometric iteration



**Figure 1.22** The Sierpinski triangle

The specific type of function that we iterate in this context is called an *iterated function system*. Some amazingly intricate images can be constructed using an iterated function system. [Figure 1.23](#) shows the Barnsely fern, which is described by a list of just four functions.



**Figure 1.23** The Barnsley fern

### 1.3 Exercises

This set of exercises will be mostly experimental. So, fire up your favorite computational environment. There are lots of potential choices but this text will generally present examples using Sage.

1. Continuing with the example of  $f(x) = x^2 - 2$  explored in [Example 1.3](#), compute ten Newton iterations for several values of  $x_0$ . Be sure to choose both positive and negative values and values that are both large and small in magnitude.
2. In the previous exercise, what happens when  $x_0 = 0$ ? Draw a graph to illustrate the situation.
3. Let  $f$  be a quadratic function that has two, distinct, real roots but that is otherwise arbitrary. Using a geometrical understanding of the real Newton's method, show why an initial seed  $x_0$  always leads to a sequence that converges to the closer of the two roots of  $f$ .

4. Let's modify Newton's original example just a little bit to consider

$$f(x) = x^3 - 2x - 2.$$

- (a) Compute the corresponding Newton's method iteration function,  $N$ .
- (b) Iterate  $N$  from the initial point  $x_0 = 0$ . What behavior do you see?

- (c) Iterate  $N$  from several initial points  $x_0$  close to zero. Now, what behavior do you see?

5. Figure 1.24 shows the graph of the function

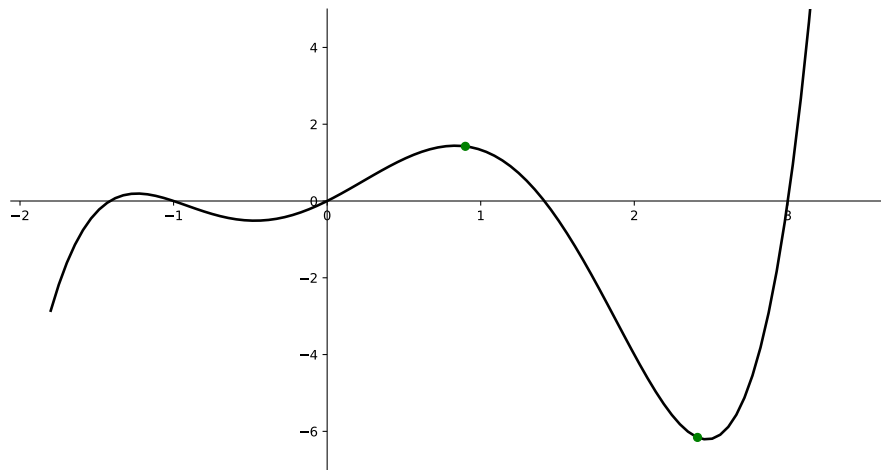
$$f(x) = \frac{1}{3}x(x+1)(x-3)(x^2-2).$$

The green dots represent points on the graph with  $x$ -coordinates that we might consider as initial seeds for Newton's method.

- (a) Suppose we start at the green dot whose  $x$  coordinate is just slightly larger than 1. To which root do you think the process will converge?
  - (b) Suppose we start at the green dot whose  $x$  coordinate is between 2 and 3. To which root do you think the process will converge?
  - (c) Find a specific value of the initial seed  $x_0$  between 2 and 3 with the property that the process converges to the smallest root of the function.
  - (d) Find a specific value of the initial seed  $x_0$  between 2 and 3 with the property that the process converges to the value 1.
6. Launch the interactive tool for generating the basins of attraction of Newton's method for polynomials here: [https://marksmath.org/visualization/complex\\_newton/](https://marksmath.org/visualization/complex_newton/).

Now, use the tool to generate images for the following polynomials and answer any additional questions that are asked.

- (a)  $f(z) = z^4 - 1$ 
  - i. What are the four roots of the function? Where do they fit into the picture?
  - ii. Click on the picture. How do you interpret the line that is drawn?
- (b)  $f(z) = (z^2 - 1)(z - 10)$ 
  - i. What initial step should you take to enter your input?
  - ii. What are the roots of the function? How could you account for this when generating the picture?
- (c)  $f(z) = z^3 - 2z - 2$ 
  - i. You should see some black regions. What's up with that?



**Figure 1.24** The graph of the function for problem 5

# Chapter 2

## The iteration of real functions

**Introduction.** In this chapter, we take a close look at discrete, real dynamics. That is, we study the iteration of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ . This concept was introduced intuitively in [Subsection 1.1.2](#).

### 2.1 Basic notions

We begin with some of the most fundamental definitions and examples. While these definitions are stated for real functions, many of them extend quite easily to other contexts.

**Definition 2.1** Let  $x_0 \in \mathbb{R}$  be an initial point and define a sequence  $(x_n)$  recursively by  $x_{n+1} = f(x_n)$ . This sequence is called *the orbit* of  $x_0$  under iteration of  $f$ .  $\diamond$

Some orbits don't move; they are fixed.

**Definition 2.2** A point  $x_0 \in \mathbb{R}$  is a *fixed point* of  $f$  if  $f(x_0) = x_0$ .  $\diamond$

Sometimes an orbit might return to the original starting point.

**Definition 2.3** Suppose that the orbit  $(x_n)$  satisfies

$$x_0 \rightarrow x_1 \rightarrow x_2 \cdots \rightarrow x_{n-1} \rightarrow x_0$$

and  $x_n = x_0$ . Such an orbit is called a *periodic orbit* and the points themselves are called *periodic points*. If  $x_k \neq x_0$  for  $k = 1, 2, \dots, n-1$ , then  $n$  is called the *period* of the orbit.  $\diamond$

Note that a fixed point is a periodic point with period one.

Sometimes, the orbit of a non-periodic point might land on a periodic orbit.

**Definition 2.4** If the zeroth term  $x_0$  of an orbit  $(x_n)$  is not periodic but  $x_n$  is periodic for some  $n$ , then  $x_0$  and its orbit are called *pre-periodic*.  $\diamond$

**Example 2.5** Let  $f(x) = 2x + 1$ . We could compute a few terms of the orbit of  $x_0 = 1$  by direct computation:

$$1 \rightarrow 3 \rightarrow 7 \rightarrow 14 \rightarrow 31 \rightarrow \cdots$$

It's not too hard to guess that a general formula for  $n^{\text{th}}$  term might be  $x_n = 2^{n+1} - 1$ . You should check this for the first few values of  $n$ .

It's easy to check that  $x_0 = -1$  is a fixed point. It seems unlikely that there are other fixed points or periodic orbits of any period.  $\square$

**Example 2.6** Let  $f(x) = x^2 - 1$ . Then zero is a periodic point and one is a pre-periodic point, as the reader may easily verify.

To find a fixed point, we can simply set  $f(x) = x$  and solve the resulting equation. In this case, we get

$$x^2 - 1 = x \text{ or } x^2 - x - 1 = 0.$$

We can then apply the quadratic formula to find that

$$x = \frac{1 \pm \sqrt{5}}{2}$$

are both fixed. □

Often, it helps to express these ideas in terms of composition of functions. We denote the  $n$  fold composition of a function with itself by  $f^n$ . That is,  $f^2 = f \circ f$  and  $f^n = f \circ f^{n-1}$ . (Be careful not to confuse this with raising a function to a power.) A more complete understanding of periodicity arises from the study of the functions  $f^n$ . For example, a point  $x_0$  has period  $n$  iff  $f^n(x_0) = x_0$  but  $f^k(x_0) \neq x_0$  for  $k = 1, 2, \dots, n-1$ .

**Checkpoint 2.7** Let  $f(x) = 2x^2 - 3x - 6$ .

1. Find the first three terms of the orbit of  $x_0 = 1$ .
2. Find all fixed points of  $f$ .
3. Write down an equation that any point of period 2 should satisfy.

## 2.2 Computer experimentation

It doesn't take long to realize that a little computer power will help develop intuition much better than, say, doing a lot of arithmetic by hand. We'll often use [Sage](#) for basic exploration of iterative dynamics. Here are a few examples.

### 2.2.1 Computing orbits

Suppose  $f(x) = \frac{1}{5}x^2 - 2x + 6$ . Here's how to compute the first five iterates of  $x_0 = 1$  under  $f$  using Sage:

```
f(x) = (1/5)*x^2 - 2*x + 6
xi = 1
for i in range(5):
    xi = f(xi)
    print(xi)
```

```
21/5
141/125
312381/78125
36639701661/30517578125
18100595397108843971421/4656612873077392578125
```

Note that the computation is exact. That is, we get rational numbers like  $21/5$ , rather than decimal approximations like  $4.2$ . Often we would prefer decimal approximations. Let's set  $xi = 1.0$  in the code block above and compute 100 iterates.



```
f(x) = (1/5)*x^2 - 2*x + 6
xi = 1.0
for i in range(100):
    xi = f(xi)
    print(xi)
```

```
4.2000000000000000
1.1280000000000000
3.9984768000000000
1.20060974402765
3.88707326343553
...
3.61804229612085
1.38196141906219
3.61804063463090
1.38196233750627
3.61803930544461
1.38196307225920
```

Sure looks like we've found a pattern! Perhaps, it's an orbit of period 2?

### 2.2.2 Finding periodic orbits

Continuing with the example of the last sub-section, suppose we'd like to find all orbits of period 2 for  $f(x) = \frac{1}{5}x^2 - 2x + 6$ . I guess we should let  $F(x) = f^2(x)$  and then find all fixed points of  $F$  or, equivalently, find all roots  $F(x) - x$ . We can automate this procedure like so:

```
f(x) = (1/5)*x^2 - 2*x + 6
F(x) = f(f(x))
(F(x)-x).roots(ring=RR)
```

```
[(1.38196601125011, 1),
 (2.37652461702020, 1),
 (3.61803398874990, 1),
 (12.6234753829798, 1)]
```

Each root is returned as a (value,multiplicity) pair. Note that the 1.38 and 3.618 agree with our prior computation; those form the orbit of period 2. The other points are fixed points. Of course, those should be the points of intersection with the line  $y = x$ . We can illustrate that with Sage too:

```
f(x) = (1/5)*x^2 - 2*x + 6
plot([x,f(x)], [x,-4,14]) +
    point([(2.37,2.37),(12.62,12.62)])
```

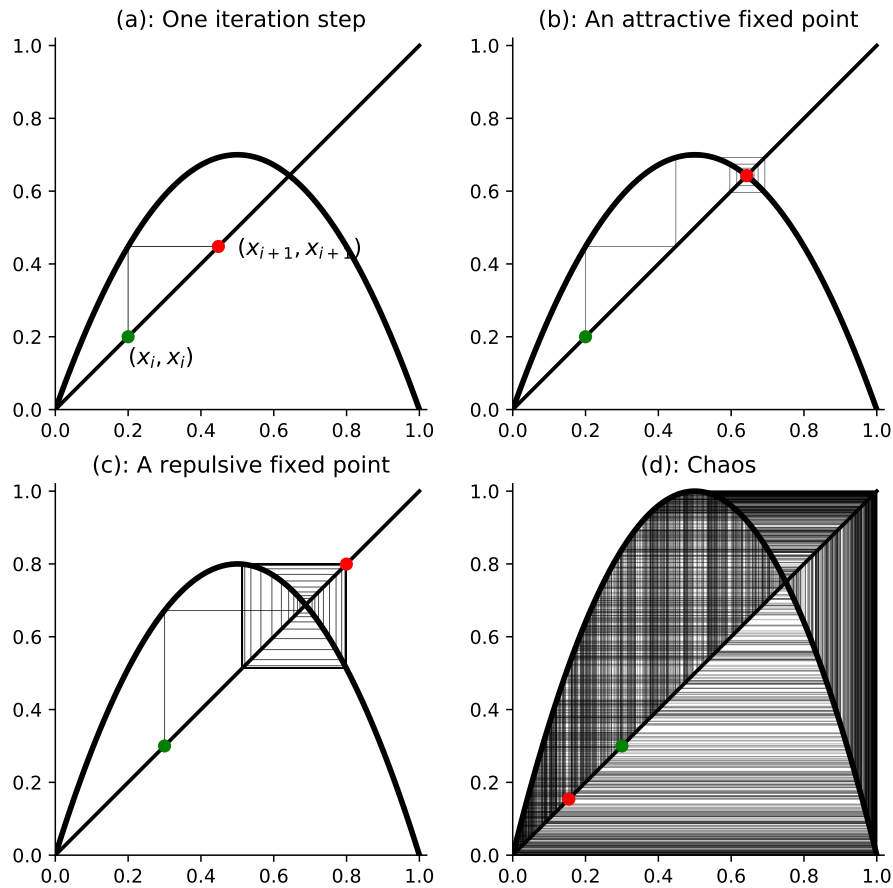
**Checkpoint 2.8 Find an orbit of period three.** Given the function  $f(x) = 7x^2 + 14x + \frac{23}{4}$ ,

1. Iterate  $f$  from the starting point  $x_0 = 1.0$  twenty times. What does the long term behavior appear to be?
2. Iterate  $f$  from the starting point  $x_0 = -1.0$  two hundred times. What does the long term behavior appear to be?
3. Write down an equation that an orbit of period 3 should satisfy and solve that equation using Sage.

## 2.3 Graphical analysis

There is an efficient geometric tool to visualize functional iteration. The basic idea is simple: Suppose we graph the function  $f$  together with the line  $y = x$ . If those two graphs intersect; that point of intersection is a fixed point. Now, suppose we're on the line at the point  $(x_i, x_i)$ . If we move vertically to the graph of the function, we preserve the  $x$  coordinate but change the  $y$  coordinate to  $f(x_i)$ . Thus, we arrive at the point  $(x_i, f(x_i)) = (x_i, x_{i+1})$ . If we then move horizontally back to the line  $y = x$  we now preserve the  $y$  coordinate but change the  $x$  coordinate so that the  $x$  and  $y$  coordinates are the same. Thus, we arrive at the point  $(x_{i+1}, x_{i+1})$ .

In summary: The process of moving vertically from a point on the line  $y = x$  to the graph of  $f$  and back to the line horizontally is a geometric representation of one application of the function  $f$ . This step is illustrated in Figure 2.9(a). Repeated application of this process represents repeated application of  $f$ , i.e. iteration. This is illustrated in Figure 2.9(b). Note that the orbit appears to be attractive.



**Figure 2.9** Some cobweb plots

It turns out that the process is quite sensitive to the slope of the function at the point of intersection. A slightly steeper function is shown in Figure 2.9(c); we notice that the fixed point now appears to be repelling. Finally, figure Figure 2.9(d) illustrates the fact that all hell can break loose.

Playing with cobweb plots is a great way to get a feel for the possibilities that arise in discrete dynamical systems. Here's an interactive tool that we'll

use for this purpose: <https://marksmath.org/visualization/cobwebs/>

**Checkpoint 2.10** Plug the function  $f(x) = \frac{1}{5}x^2 - 2x + 6$  that we played with in Section 2.2 into the [cobweb tool](#). Can you see why we couldn't help but find the periodic behavior?

## 2.4 Classification of fixed points

The cobweb plots in the previous section illustrate that the slope of the function at the point where it crosses the fixed point might play a role in the behavior of the iterates near that fixed point. We explore that further here. First, we explore the simplest situation - functions with constant slope.

**Example 2.11 Linear iteration.** Suppose that  $f$  is a linear function:  $f(x) = ax$ . It's easy to see that the origin  $x = 0$  is a fixed point of  $f$ . Show that any non-zero initial point  $x_0$  moves away from the origin under the iteration of  $f$  whenever  $|a| > 1$  but moves towards the origin under iteration of  $f$  if  $|a| < 1$ .

**Solution.** This is easy, once we recognize that there is a closed form for the  $n^{\text{th}}$  iterate of  $f$ , namely  $f^n(x) = a^n x$ . Note that cobweb plots for these functions are shown in Figure 2.12.

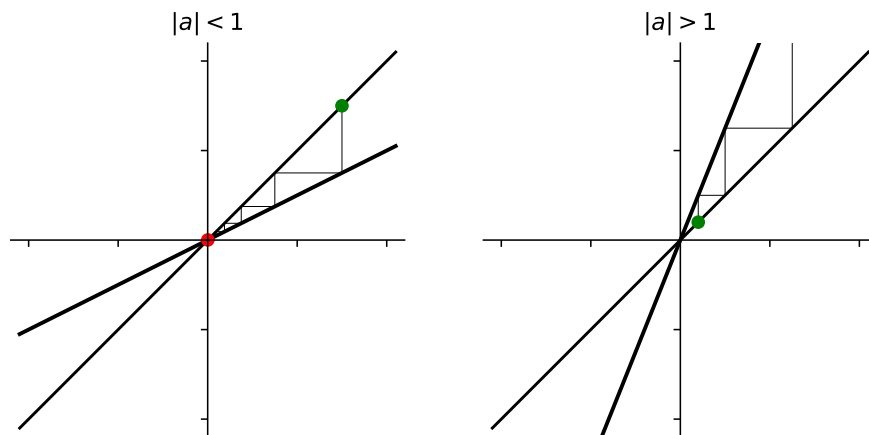


Figure 2.12 Some linear cobweb plots

□

**Checkpoint 2.13 Affine iteration.** Suppose, that  $f$  is an *affine function*, which just means that it has the form  $f(x) = ax + b$ , where  $a \neq 0$ . Suppose also that  $x_0 \in \mathbb{R}$  and let's consider the iterates  $x_{n+1} = f(x_n)$

1. Show that  $f$  has a unique fixed point iff  $a \neq 1$ . What if  $a = 1$ ?
2. Suppose that  $|a| < 1$ . Show that the sequence of iterates converges to the fixed point of  $f$ .
3. Suppose that  $|a| > 1$ . Show that the sequence of iterates diverges.
4. What happens if  $a = -1$ ?

Example [Example 2.11](#) and exercise [Checkpoint 2.13](#) together classify the dynamical behavior of first order polynomials completely and show that their behavior is fairly simple. For that reason, we focus on polynomials of degree two and higher. Already in the quadratic case, we can find much more complicated

and interesting behavior. Motivated by the behavior we see in linear and affine functions, we make the following definition.

**Definition 2.14** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable and suppose that  $x_0 \in \mathbb{R}$  is a fixed point of  $f$ . Then we classify  $x_0$  as

1. *attractive*, if  $0 < |f'(x_0)| < 1$ ,
2. *super-attractive*, if  $f'(x_0) = 0$ ,
3. *repulsive* or *repelling*, if  $|f'(x_0)| > 1$ , or
4. *neutral*, if  $|f'(x_0)| = 1$ ,

The number,  $f'(x_0)$  is called the *multiplier* for the fixed point. If, in the attractive case, the multiplier is zero, we say that  $x_0$  is *super-attractive*.  $\diamond$

The following theorem justifies this notation.

**Theorem 2.15** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable and suppose that  $x_0 \in \mathbb{R}$  is a fixed point of  $f$ .

1. If  $x_0$  is an attractive or super-attractive fixed point for  $f$ , then there is an  $\varepsilon > 0$  such that the orbit of  $x$  under iteration of  $f$  tends to  $x_0$  for every  $x$  such that  $|x - x_0| < \varepsilon$ .
2. If  $x_0$  is a repelling fixed point for  $f$ , then there is an  $\varepsilon > 0$  such that the orbit of  $x$  under iteration of  $f$  tends (initially) away from  $x_0$  for every  $x$  such that  $|x - x_0| < \varepsilon$ .

*Proof.* We prove part one; the second part is similar. Since  $|f'(x_0)| < 1$  and  $f'$  is continuous, we may choose an  $\varepsilon > 0$  and a positive number  $r < 1$  such that  $|f'(x)| < r$  for all  $x$  such that  $|x - x_0| < \varepsilon$ . Then, given  $x$  such that  $|x - x_0| < \varepsilon$ , we can apply the Mean Value Theorem to obtain a number  $c$  such that

$$|f(x) - x_0| = |f(x) - f(x_0)| = |f'(c)||x - x_0| \leq r\varepsilon.$$

By induction, we can show that

$$|f^n(x) - x_0| \leq r^n \varepsilon.$$

The result follows, since  $r^n \varepsilon \rightarrow 0$  as  $n \rightarrow \infty$ .  $\blacksquare$

From the proof, we see that  $x_n \rightarrow x_0$  exponentially and that the magnitude of  $|f'(x_0)|$  dictates the base of that exponential. When  $f'(x_0) = 0$ , the rate is faster than exponential.

**Example 2.16** The function  $f(x) = 4.8x^2(1 - x)$  is graphed in [Figure 2.17](#), along with the line  $y = x$ . The points of intersection are fixed points and, from left to right, they are super-attractive, repulsive, and attractive. The reader should consider the appearance of a cobweb plot for initial values starting near each of those fixed points.

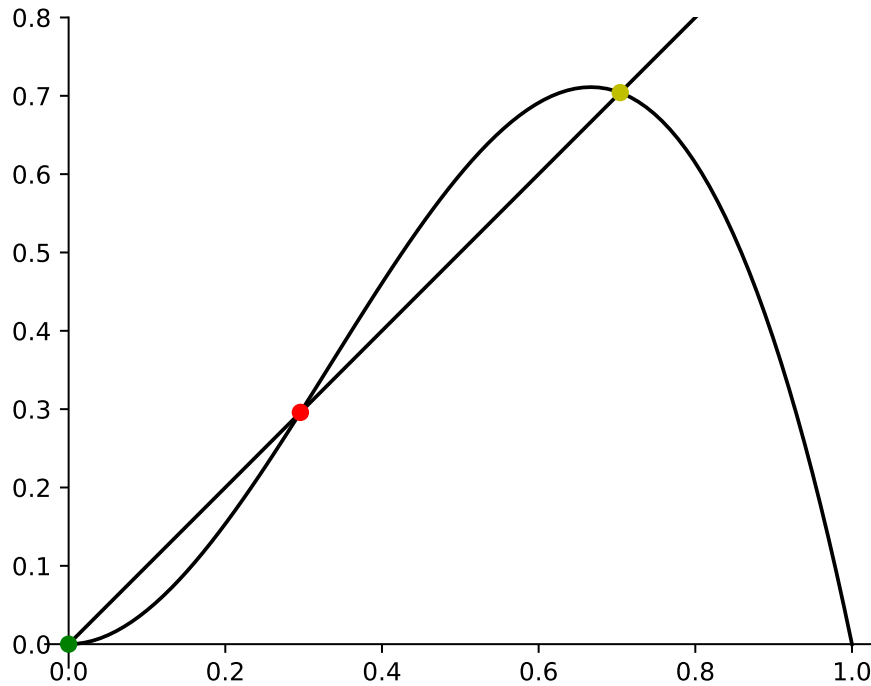


Figure 2.17 Three types of fixed points

□

The behavior of iterates near a neutral fixed point can be more varied.

**Checkpoint 2.18 Dynamical behavior near neutral fixed points.** For each of the following scenarios, find an example of a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and a fixed point  $x_0$  of  $f$  satisfying that scenario.

1. There is an  $\varepsilon > 0$  such that the orbit of  $x$  tends to  $x_0$  for all  $x$  such that  $|x - x_0| < \varepsilon$ .
2. There is an  $\varepsilon > 0$  such that the orbit of  $x$  tends initially away from  $x_0$  for all  $x$  such that  $|x - x_0| < \varepsilon$ .
3. There is an  $\varepsilon > 0$  such that the orbit of  $x$  tends to  $x_0$  for all  $x$  such that  $0 < x - x_0 < \varepsilon$  but the orbit of  $x$  tends initially away from  $x_0$  for all  $x$  such that  $0 < x_0 - x < \varepsilon$ .
4. There is an  $\varepsilon > 0$  such that the orbit of  $x$  tends to  $x_0$  for all  $x$  such that  $0 < x - x_0 < \varepsilon$  but the orbit of  $x$  tends initially away from  $x_0$  for all  $x$  such that  $0 < x_0 - x < \varepsilon$ .

## 2.5 Classification of periodic orbits

As mentioned right after [the example on periodicity 2.6](#), a periodic point for  $f$  of period  $n$  is a fixed point of  $f^n$ . Treating the points of a periodic orbit this way allows us to extend the classification as fixed points to periodic orbits.

**Definition 2.19** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable and suppose that  $x_0 \in \mathbb{R}$  is a periodic point of  $f$  with period  $n$ . Let  $F = f^n$ . We classify  $x_0$  and its orbit as

1. *attractive*, if  $|F'(x_0)| < 1$ ,

2. *super-attractive*, if  $F'(x_0) = 0$ ,
3. *repulsive or repelling*, if  $|F'(x_0)| > 1$ , or
4. *neutral*, if  $|F'(x_0)| = 1$ ,

The number  $F'(x_0)$  is called the *multiplier* of the orbit. If, in the attractive case, the multiplier is zero, we say that the orbit is *super-attractive*.  $\diamond$

There is a nice characterization of the multiplier of an orbit that allows us to compute it without explicitly computing a formula for  $f^n$ .

**Lemma 2.20** *Suppose that*

$$x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_{n-1} \rightarrow x_0$$

*is an orbit of period  $n$  for  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Then the multiplier of the orbit is*

$$f'(x_0)f'(x_1) \cdots f'(x_{n-1}).$$

*Proof.* First note that for an  $n = 2$ , we can apply the chain rule to obtain

$$\frac{d}{dx}f^2(x) = \frac{d}{dx}f(f(x)) = f'(f(x))f'(x).$$

Thus, if  $x_0 \rightarrow x_1 \rightarrow x_0$  is an orbit of period two and we evaluate that equation at  $x_0$ , we obtain

$$\left. \frac{d}{dx}f^2(x) \right|_{x=x_0} = f'(x_1)f'(x_0).$$

The result for orbits longer than two can be proven by induction, since

$$\frac{d}{dx}f^n(x) = \frac{d}{dx}f(f^{n-1}(x)) = f'(f^{n-1}(x))\frac{d}{dx}f^{n-1}(x).$$

■

Note that the only way the product in [Lemma 2.20](#) is zero, is if one of the terms is zero. This yields the following corollary.

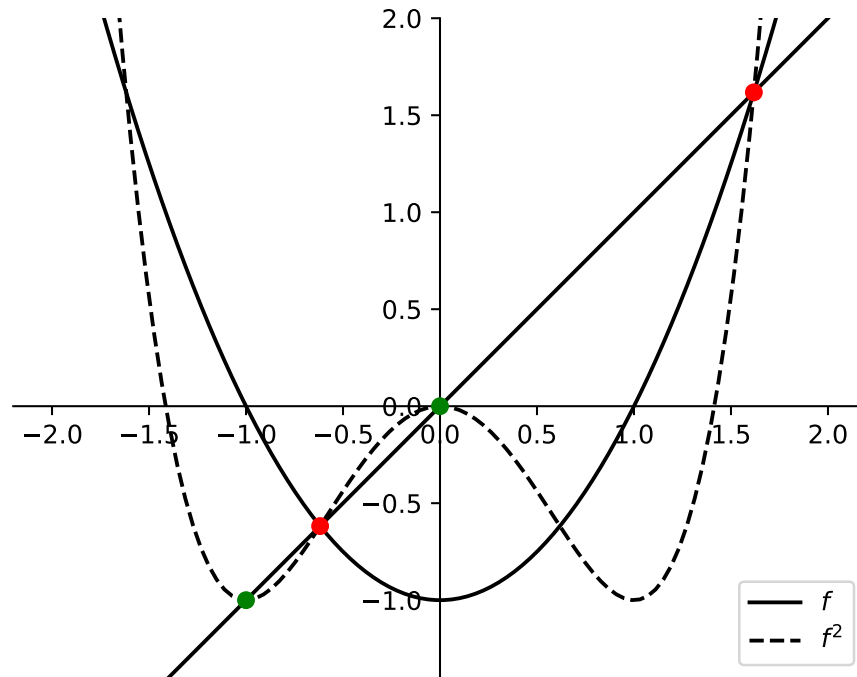
**Corollary 2.21** *A periodic orbit is super-attracting if and only if it contains a critical point.*

**Example 2.22** Let  $f(x) = x^2 - 1$ . Note that  $f(0) = -1$  and  $f(-1) = 0$  so that  $0 \rightarrow 1 \rightarrow 0$  forms an orbit of period 2. To see if this orbit is attractive, we examine

$$F(x) = f(f(x)) = (x^2 - 1)^2 - 1 = x^4 - x^2.$$

Note that  $F'(0) = 0$  and  $F'(-1) = 0$ ; thus, the orbit is super-attractive.

The plots of  $f$  and  $f^2$ , together with  $y = x$ , are shown in [Figure 2.23](#). Note that  $f$  has two fixed points shown in red. They can be found by solving the equation  $x^2 - 1 = x$  and they are both repulsive under iteration of  $f$ . The two super-attractive orbits of  $f^2$  are shown in green.



**Figure 2.23** An attractive orbit of period two

□

## 2.6 Parameterized families of functions

Rather than explore the behavior of a single function at a time, we can introduce a parameter and explore the range of behavior that arises in a whole family of functions. Two important examples are

1. *The quadratic family:*  $f_c(x) = x^2 + c$
2. *The logistic family:*  $f_\lambda(x) = \lambda x(1 - x)$

The cobweb plots shown back in [Figure 2.9](#) are all chosen from the logistic family with  $\lambda = 2.8$ ,  $\lambda = 3.2$ , and  $\lambda = 4$ . Even in those three pictures with graphs that look so very similar, we see three different types of behavior: an attractive fixed point, an attractive orbit of period two, and chaos (which can be given a very technical meaning).

[Figure 2.24](#) shows some cobweb plots for the quadratic family of functions. Note that the behavior we see is very similar to the behavior we see for the logistic family - a fact that will become more understandable once we study conjugacy in [Section 2.7](#)

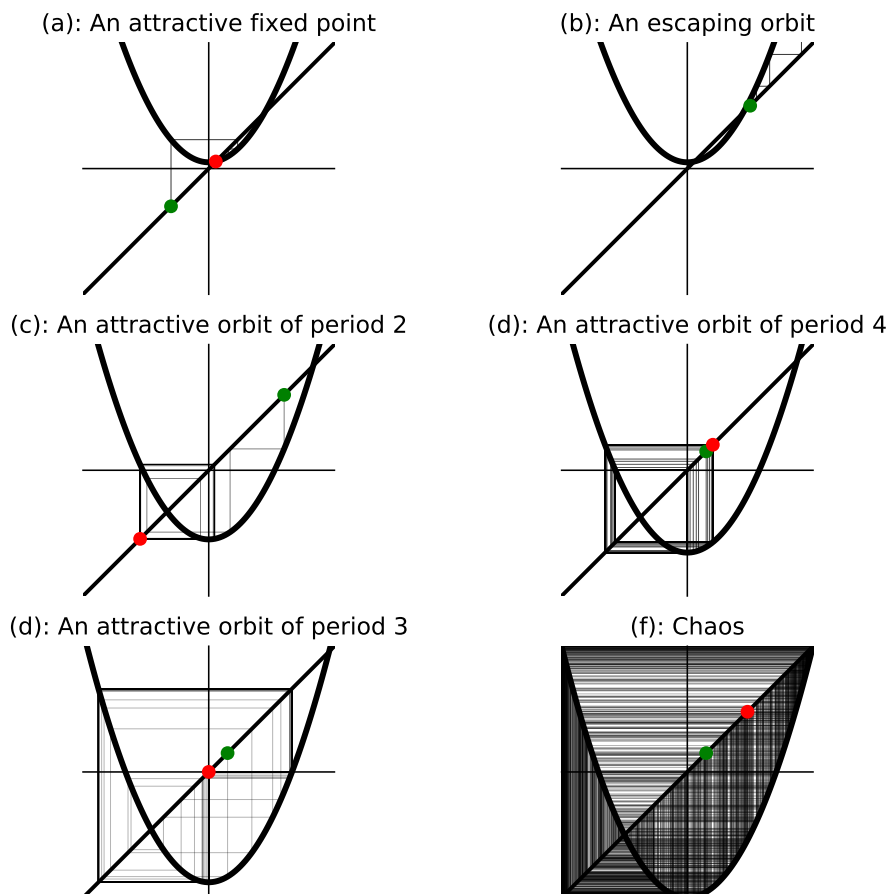


Figure 2.24 Some cobweb plots for the quadratic family

### 2.6.1 The bifurcation diagram

A fabulous illustration of the types of behavior that can arise in a family of functions indexed by a single parameter and each with a single critical point can be generated as follows: For each value of the parameter, compute a large number points of the orbit of the critical point (maybe 1000 iterates). Since we're interested in long term behavior, rather than any transient behavior, discard the first few iterates (maybe 100). Then, plot the remaining points in a vertical column at the horizontal position indicated by the parameter.

The orbit of a critical point is called a *critical orbit* and its importance is due to the following theorem.

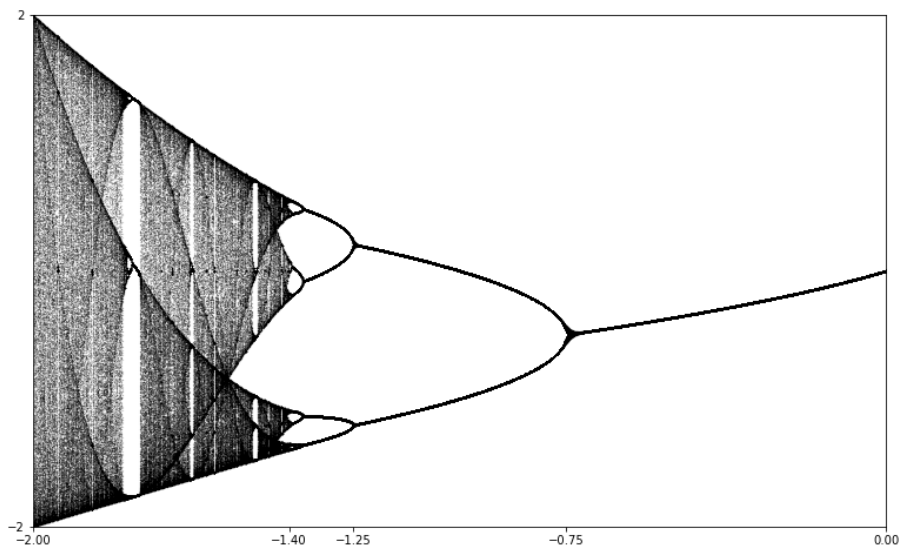
**Theorem 2.25** *If  $f : \mathbb{C} \rightarrow \mathbb{C}$  has an attractive or super-attractive orbit, then that orbit must attract at least one critical point.*

Note that this is really a theorem of complex dynamics. There is an analogous statement for real dynamics but it's a bit more complicated and its proof takes us a bit farther astray than we want. This is a great example of complex analysis being, in some ways, more elegant than real analysis.

Regardless, the theorem has important implications for real iteration. For example, a polynomial of degree  $n$  can have at most  $n - 1$  attractive orbits. Furthermore, if all the critical points happen to be real, we can find all the attractive behavior by simply iterating from the critical points. If we do this



systematically for the quadratic family, plotting the columns to generate the bifurcation diagram, we get [Figure 2.26](#)



**Figure 2.26** The bifurcation diagram for the quadratic family

We can interpret this diagram as follows:

- For  $-0.75 < c < 0$ , there is an attractive fixed point.
- For  $-1.25 < c < -0.75$ , there is an attractive orbit of period 2.
- As  $c$  passes from just above  $-0.75$  to just below  $-0.75$ , the dynamics of  $f_c$  undergo a *bifurcation*.
- For  $c$  just a little less than  $-1.25$ , there is an attractive orbit of period four. This orbit bifurcates soon into an attractive orbit of period 8. It appears that this behavior continues as  $c$  decreases.
- For  $c$  somewhere around  $c \approx -1.4$ , the period doubling appears to stop and we get more complicated behavior.

Generally, a bifurcation occurs at a parameter value  $c = c_0$  if the global dynamical behavior of the function  $f_c$  undergoes some qualitative change as  $c$  passes through  $c_0$ . There are number of different types of bifurcations that can occur, depending on the nature of the qualitative behavior under consideration. The bifurcations that are evident in [Figure 2.26](#) in the range  $-1.4 < c < 0$  are called *period doubling bifurcations*.

### 2.6.2 The period doubling cascade

Let's work towards a deeper, theoretical understanding of the period doubling that we see in the bifurcation diagram of [Figure 2.26](#). Again, we are dealing with the family of functions  $f_c(x) = x^2 + c$ . For  $c$  just a bit larger than  $-0.75$  it appears that we have an attractive fixed point while, for  $c$  just a bit smaller than  $-0.75$ , it appears that we have an attracting orbit of period two. Why, exactly does this happen?

First, let's explore the fixed points of  $f_c$ ; we can find them by solving  $f_c(x) = x$ :

$$x^2 + c = x \iff x^2 - x + c = 0.$$

Applying the quadratic formula, we find

$$x = \frac{1 \pm \sqrt{1 - 4c}}{2}.$$

For  $c < 1/4$ , we have two real fixed points but a glance at the graphs from [Figure 2.24](#) shows that it's the smaller of these two fixed points we're interested in. Of course,  $f'(x) = 2x$ , so the value of the derivative at the smaller fixed point is  $1 - \sqrt{1 - 4c}$ . Plugging  $c = -3/4$  into this formula, we find that this is  $-1$ . For  $c$  slightly larger than  $-3/4$ , this is bigger than  $-1$  and for  $c$  slightly smaller than  $-3/4$ , this is smaller than  $-1$ . This explains why we have an attractive fixed point for  $c$  slightly larger than  $-3/4$  that is no longer attractive once  $c$  passes below  $-3/4$ .

Now, we ask - why does the attractive orbit of period two appear as the attractive fixed point disappears? To see this, we consider the function

$$F_c(x) = f_c \circ f_c(x) = (x^2 + c)^2 + c = x^4 + 2cx^2 + (c^2 + c).$$

We are interested in the fixed points, thus we must solve

$$x^4 + 2cx^2 + (c^2 + c) = x \text{ or } x^4 + 2cx^2 - x + (c^2 + c) = 0. \quad (2.1)$$

Here is an observation that helps us factor this polynomial: Any point that is fixed by  $f_c$  must also be fixed by  $F_c$ . Thus, we expect  $x^2 + c - x$  to be a factor of the polynomial in (2.1). Using this, we find that

$$x^4 + 2cx^2 - x + (c^2 + c) = (x^2 - x + c)(x^2 + x + c + 1).$$

We can then apply the quadratic formula to get the two new fixed points of  $F_c$ , namely

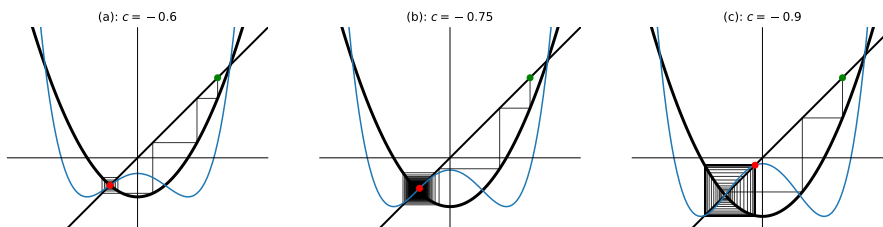
$$x = \frac{-1 \pm \sqrt{1 - 4(c+1)}}{2} = \frac{-1 \pm \sqrt{-(3+4c)}}{2}.$$

These two points form an orbit of period two for  $f_c$ . Since  $f'_c(x) = 2x$  we can multiply those points by two and multiply the results to get the multiplier for the orbit. The result is:

$$(-1 + \sqrt{-(3+4c)})(-1 - \sqrt{-(3+4c)}) = 4 + 4c.$$

When  $c = -3/4$ , the multiplier is 1. For  $c$  a little less than  $-3/4$ , the multiplier is a little less than one. Hence the orbit has become attractive.

A nice way to visualize this is to plot  $f_c^2$  together with  $f_c$  and  $y = x$  on the same set of axes for a few different choices of  $c$ . This is shown in [Figure 2.27](#) where we can see exactly how the fixed point went from attractive to repulsive while an attractive orbit of period two showed up as  $c$  passed below  $-0.75$ .



**Figure 2.27** Bifurcation

Note that our [cobweb tool](#) allows you to plot iterates of  $f$  on the plot to assist in this type of analysis.

**Checkpoint 2.28** Consider the logistic family  $f_\lambda(x) = \lambda x(1 - x)$ .

1. Using our [cobweb tool](#)

## 2.7 Conjugacy

[Figure 2.9](#) and [Figure 2.24](#) show that the iterative behavior of the logistic family and the quadratic family are very similar. In a sense, they are identical. We make that notion precise in this section.

**Definition 2.29 Conjugacy.** Let  $S$  and  $T$  be sets and suppose that  $f : S \rightarrow S$  and  $g : T \rightarrow T$ . We say that  $f$  is *semi-conjugate* to  $g$  if there is a surjective function  $\varphi : T \rightarrow S$  such that

$$f \circ \varphi = \varphi \circ g.$$

The function  $\varphi$  is called a *semi-conjugacy*. In the case that  $\varphi$  is bijective, then we say that  $\varphi$  is a conjugacy and that  $f$  and  $g$  are conjugate.  $\diamond$

A geo-symbolic way to remember the semi-conjugation formula is in the form of a commutative diagram:

$$\begin{array}{ccc} T & \xrightarrow{g} & T \\ \downarrow \varphi & & \downarrow \varphi \\ S & \xrightarrow{f} & S \end{array}$$

The formula states if you follow the arrows from the upper left to the lower right in either direction, you get the same result.

An immediate consequence of the definition of conjugacy is

$$f^2 \circ \varphi = f \circ f \circ \varphi = f \circ \varphi \circ g = \varphi \circ g \circ g = \varphi \circ g^2$$

and, by induction

$$f^n \circ \varphi = \varphi \circ g^n.$$

As a result, if  $(t_i)$  is an orbit of  $g$ , then  $(\varphi(t_i))$  is an orbit of  $f$ .

Generally, the nicer  $\varphi$  is, the closer the relationship between the dynamics of  $f$  and the dynamics of  $g$ . If  $\varphi$  is bijective, then the relationship is quite close. If  $\varphi$  is continuous with continuous inverse, then topological properties of the orbits will be preserved. If  $S$  and  $T$  are sets of real or complex numbers and  $\varphi(x) = ax + b$ , then an orbit of one function will be geometrically similar to an orbit of the other. The dynamical systems are truly identical, up to a scaling.

**Example 2.30** Show that  $f(x) = x^2 - 1$  is conjugate to  $g(x) = \frac{1}{2}x^2 + 2x - 2$  via the conjugacy  $\varphi(x) = \frac{1}{2}x + 1$ .

**Solution.** We simply compute

$$\begin{aligned} f(\varphi(x)) &= \left(\frac{1}{2}x + 1\right)^2 - 1 = \frac{1}{4}x^2 + x \\ \varphi(g(x)) &= \frac{1}{2} \left(\frac{1}{2}x^2 + 2x - 2\right) + 1 = \frac{1}{4}x^2 + x. \end{aligned}$$

[Figure 2.31](#) illustrates the similarity between the two functions.

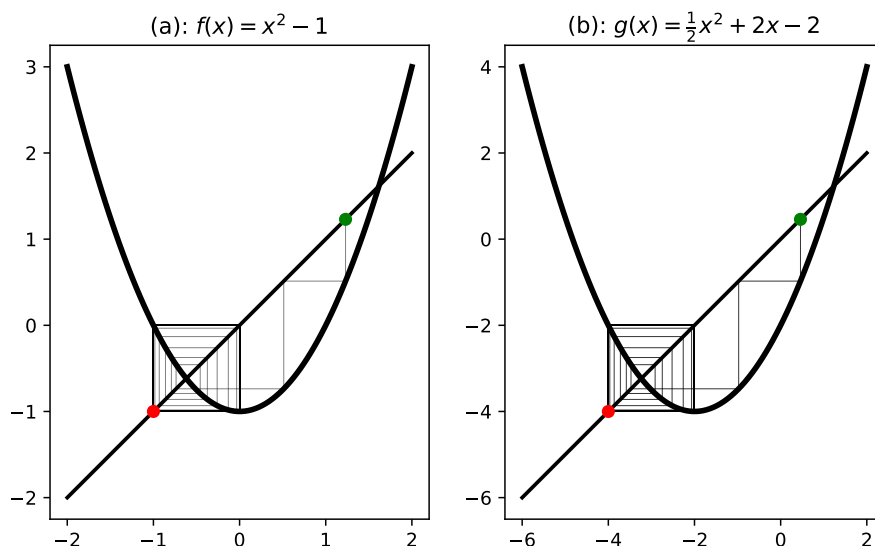


Figure 2.31 Cobweb plots for conjugate functions

□

If you suspect that  $f$  is conjugate to  $g$  via a conjugacy of the form  $\varphi(x) = ax + b$ , then you can find that conjugacy by setting  $f(\varphi(x)) = \varphi(g(x))$ . If you compare coefficients, you should get a system of equations that you can solve for  $a$  and  $b$  yielding the conjugacy.

**Checkpoint 2.32** Find a conjugacy of the form  $\varphi(x) = ax + b$  from  $f(x) = x^2 - 2$  to  $g(x) = 4x(1 - x)$ .

Exercise [Checkpoint 2.32](#) can be generalized. In fact, the quadratic family for  $-2 \leq c \leq 1/4$  is identical to the logistic family for  $1 \leq \lambda \leq 4$ .

**Checkpoint 2.33** Show that  $f(x) = x^2 + (2\lambda - \lambda^2)/4$  is conjugate to  $g(x) = \lambda x(1 - x)$  via the conjugacy  $\varphi(x) = -\lambda x + \lambda/2$ .

## 2.8 The doubling map and chaos

A glance at the cobweb plots of  $f(x) = x^2 - 2$  and  $g(x) = 4x(1 - x)$  shows that they both exhibit very complicated behavior. In fact, they are chaotic in a perfectly quantitative sense. In this section, we'll introduce the doubling map, which is (in a sense) the prototypical chaotic map. After seeing why it's chaotic, we'll show that it's conjugate to  $f(x) = x^2 - 2$ , implying that it too is chaotic.

### 2.8.1 The doubling map

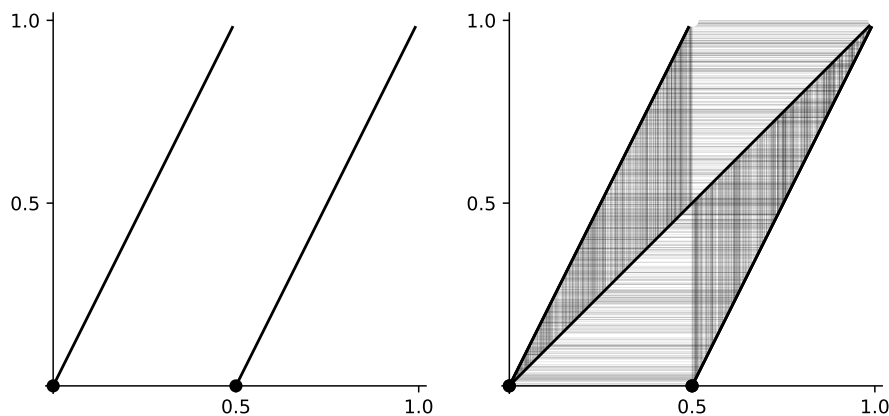
Let  $H$  denote the half-open, half-closed unit interval:

$$H = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}.$$

The doubling map  $d$  is the function  $d : H \rightarrow H$  defined by

$$d(x) = 2x \bmod 1.$$

A graph of the doubling map together with a typical cobweb plot starting at an irrational number is shown in [Figure 2.34](#)



**Figure 2.34** The doubling map

As it turns out, the doubling map is particularly easy to analyze if we consider its effect on the binary representation of a number. Suppose that  $x \in H$  has binary representation

$$x = 0_2 b_1 b_2 b_3 b_4 b_5 \cdots,$$

where each  $b_i$  is a zero or a one. (In computer parlance, the  $b_i$ s are called the *bits* of the number.)

Now, the effect of  $d$  on the binary representation of  $x$  is simple:

$$d(x) = 0_2 b_2 b_3 b_4 b_5 \cdots.$$

That is, the effect of  $d$  is to simply shift the bits of  $x$  to the left, discarding the bit that shifted into the ones place.

Of course, some numbers have multiple binary representations. For example,

$$0_2 1 = 0_2 0\bar{1} = \frac{1}{2},$$

The doubling map, though, is defined independently of the binary representation of the number. Thus, we would expect that the shift operation, when applied to different representations of the same number, should lead to the same result. For example, the shift operation when applied to the two representations of  $1/2$  above yield.

$$0_2 \bar{0} = 0_2 \bar{1} = 0 \pmod{1}.$$

This observation makes it very easy to find orbits with specific properties. Suppose, for example, we want an orbit of period 3. Simply pick (almost) any number of the form

$$x = 0_2 \overline{b_1 b_2 b_3}$$

The only caveat is that we can't have all  $b_i$ s the same for that would lead to either zero (which is fixed) or one (which is not in  $H$ ). As a concrete example,

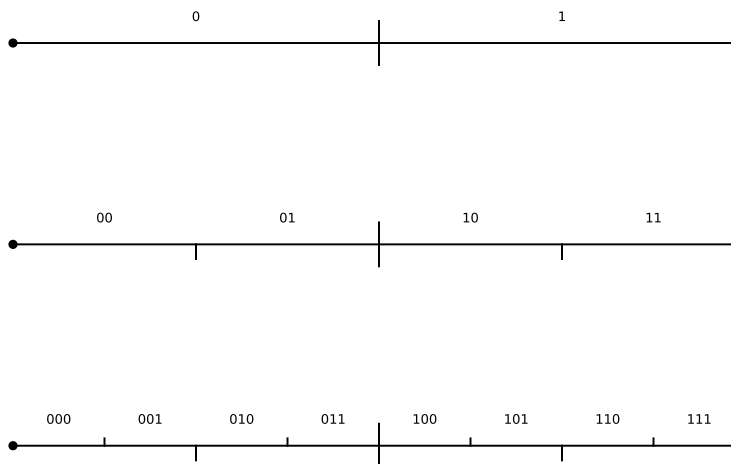
$$x = 0_2 \overline{001} = \sum_{k=1}^{\infty} \frac{1}{8^k} = \frac{1}{7}$$

has period 3. In fact, it's easy to verify that

$$\frac{1}{7} \rightarrow \frac{2}{7} \rightarrow \frac{4}{7} \rightarrow \frac{8}{7} = \frac{1}{7} \pmod{1}$$

under the doubling map.

Another nice feature of this representation is that there is a simple correspondence between the binary expansion of a number and its position in the unit interval. Every number with a binary expansion starting with a zero lies in the left half of the unit interval, while every number starting with a one lies in the right half. The first two bits of a number specify in which quarter of the interval the number lies; the first three bits specify in which eighth of the unit interval the number lies, as shown in [Figure 2.35](#)



**Figure 2.35** Dyadic intervals

More generally, given  $n \in \mathbb{N}$ , we can break the unit interval up into  $n$  pieces with length  $1/2^n$  and endpoints  $i/2^n$  for  $i = 0, 1, \dots, 2^n$ . These are called *dyadic intervals* and their endpoints (number of the form  $1/2^n$ ) are called *dyadic rationals*. The first  $n$  bits of a number specify in which  $n^{\text{th}}$  level dyadic interval that number lies. In fact, the left hand endpoint of a dyadic interval has a terminating binary expansion which tells you exactly the first  $n$  bits of all the points in that interval.

Now, suppose that

$$x_1 = 0_2 b_1 b_2 \cdots b_n b_{n+1} b'_{n+2} \cdots \quad \text{and} \quad x_2 = 0_2 b_1 b_2 \cdots b_n b'_{n+1} b'_{n+2} \cdots.$$

Thus, the binary expansions of  $x_1$  and  $x_2$  agree up to at least the  $n^{\text{th}}$  spot but potentially disagree after that. Then, our geometric understanding of dyadic intervals allows us to easily see that,

$$|x_1 - x_2| \leq \frac{1}{2^n}.$$

Of course, there's also a simple algebraic proof of this fact, based on the fact that the bits cancel for  $k \leq n$

$$\begin{aligned} |x_1 - x_2| &= \left| \sum_{k=n+1}^{\infty} \frac{b_k - b'_k}{2^k} \right| \\ &\leq \sum_{k=n+1}^{\infty} \frac{|b_k - b'_k|}{2^k} \leq \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}. \end{aligned}$$

### 2.8.2 Chaos

We can now prove three claims about the doubling map that, together, assert that the doubling map displays some of the essential features of chaos. First, we'll need to state and prove a lemma.

**Lemma 2.36** *Suppose that*

$$x = 0_2 b_1 b_2 b_3 \cdots \text{ and } y = 0_2 b'_1 b'_2 b'_3 \cdots$$

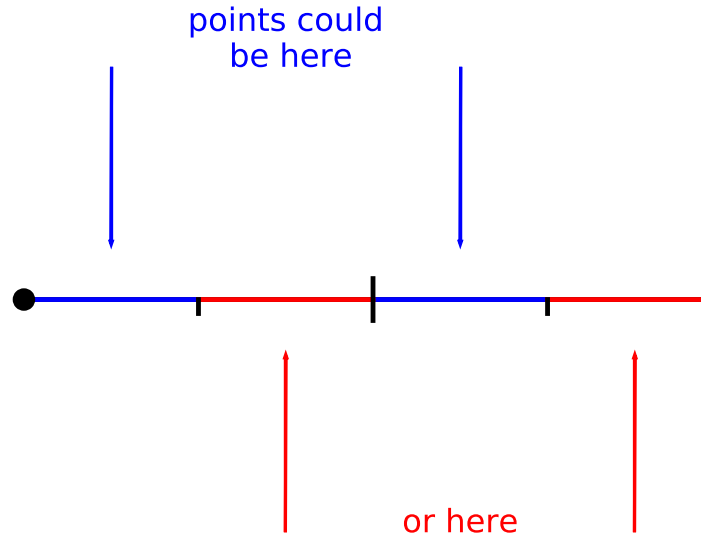
*are elements of  $H$  that satisfy  $b_1 \neq b'_1$  but  $b_2 = b'_2$ . Then  $|x - y| \geq 1/4$ .*

*Proof.* Computing the difference using the binary representations, taking into account that the terms disagree in the first spot and agree in the second, and finally applying the reverse triangle inequality, we get

$$\begin{aligned} |x - y| &= \left| \sum_{i=1}^{\infty} \frac{b_i - b'_i}{2^i} \right| = \left| \pm \frac{1}{2} + \sum_{i=3}^{\infty} \frac{b_i - b'_i}{2^i} \right| \\ &\geq \left| \pm \frac{1}{2} \right| - \left| \sum_{i=3}^{\infty} \frac{b_i - b'_i}{2^i} \right| \geq \left| \frac{1}{2} - \frac{1}{4} \right| = \frac{1}{4}. \end{aligned}$$

■

A geometric interpretation of this lemma is as follows. The fact that the two points disagree in the first spot means that they cannot lie in the same half of  $H$ . The fact that they do agree in the second spot means that they lie in the same quarter relative to their half, as shown in [Figure 2.37](#). Clearly, any two such points cannot be within  $1/4$  of one another.



**Figure 2.37** Possible positions of points in lemma [Lemma 2.36](#)

**Claim 2.38 Sensitive dependence on initial conditions.** *For every  $x \in H$  and for every  $\varepsilon > 0$ , there is some  $y \in H$  and an  $n \in \mathbb{N}$  such that  $|x - y| < \varepsilon$  yet  $|d^n(x) - d^n(y)| \geq 1/4$ .*

*Proof.* Choose  $n \in \mathbb{N}$  large enough so that  $1/2^n < \varepsilon$ . Now suppose that  $x \in H$  has binary expansion

$$x = 0_2 b_1 b_2 \cdots b_n b_{n+1} b_{n+2} \cdots .$$

Define  $y \in H$  so that

$$y = 0_2 b_1 b_2 \cdots b_n (1 - b_{n+1}) b_{n+2} \cdots .$$

That is, the bits of  $y$  agree with those of  $x$  in the first  $n$  spots, disagree with  $x$  in the  $(n+1)^{\text{st}}$  spot, and finally agree with  $x$  again in the  $(n+2)^{\text{nd}}$  spot.

Then, the numbers  $d^n(x)$  and  $d^n(y)$  satisfy the hypotheses of lemma [Lemma 2.36](#), thus  $|d^n(x) - d^n(y)| \geq 1/4$ . ■

**Claim 2.39 Denseness of periodic orbits.** *For every open interval  $I \subset H$ , there is some periodic orbit with an element in  $I$ .*

*Proof.* Let  $x \in I$  and choose  $n \in \mathbb{N}$  large enough so that

$$(x, x + 1/2^n) \subset I.$$

Now suppose that

$$x = 0_2 b_1 b_2 \cdots b_n \cdots .$$

Then,

$$\hat{x} = 0_2 \overline{b_1 b_2 \cdots b_n}$$

is a periodic point in  $I$ . ■

**Claim 2.40 A dense orbit.** *There is a point  $x \in H$  with the property that, for every open interval  $I \subset H$ , there is some iterate of  $x$  in  $I$ .*

*Proof.* We'll define  $x$  by specifying its binary expansion. We begin by writing down all possible *finite* binary strings:

$$0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \dots$$

We then concatenate these to obtain the binary representation of  $x$

$$x = 0_2 010011011000001010011100101110111 \dots$$

Now, let  $I \subset H$  be an open interval. We claim that there is some iterate of  $x$  in  $I$ . To see that, let  $L$  denote the length of  $I$  and choose  $n \in \mathbb{N}$  large enough so that

$$\frac{1}{2^n} < \frac{1}{2} L.$$

Let  $i$  be the smallest integer such that  $i/2^n \in I$ . Note that we must also have  $(i+1)/2^n \in I$ . Thus, the dyadic interval  $[i/2^n, (i+1)/2^n)$  is wholly contained in  $I$  and the first  $n$  bits of every point in that interval agree with  $i/2^n$ . So, let

$$\frac{i}{2^n} = 0_2 b_1 b_2 \cdots b_n$$

and note that, by construction, the string  $b_1 b_2 \cdots b_n$  appears somewhere in the binary expansion of  $x$ . Thus, we can apply the doubling function to the point  $x$  some number, say  $m$ , times to obtain

$$d^m(x) = 0_2 b_1 b_2 \cdots b_n \cdots .$$

The number  $d^m(x)$  is then an iterate of  $x$  that lies in  $I$ . ■



While there is no truly universally accepted definition of chaos, claims [Claim 2.38](#), [Claim 2.39](#), and [Claim 2.40](#) are generally agreed to express some of the essential features of chaos. We might think of them as representing:

- Instability,
- Structure, and
- Indecomposability

### 2.8.3 A chaotic quadratic

Let  $f(x) = x^2 - 2$ . We now show that  $f$  is semi-conjugate to the doubling map  $d$  under the semi-conjugacy  $\varphi(x) = 2 \cos(2\pi x)$ . As a result,  $\varphi$  maps all the orbit types that  $d$  has to an orbit of  $f$  with similar properties. Thus,  $f$  is chaotic.

**Claim 2.41** *The map  $f(x) = x^2 - 2$  is semi-conjugate to the doubling map  $d(x) = 2x \bmod 1$  under the semi-conjugacy  $\varphi(x) = 2 \cos(2\pi x)$ .*

*Proof.* We must simply show that  $f \circ \varphi = \varphi \circ d$ , so let's compute. First,

$$f(\varphi(x)) = 2(2 \cos(2\pi x))^2 - 2 = 4 \cos^2(2\pi x) - 2.$$

Well, that was easy. The next part is a little trickier - we just need to apply a couple of trig identities and use the fact that we can drop the mods inside the squared trig functions due to the symmetries of those functions.

$$\begin{aligned} \varphi(d(x)) &= 2 \cos(2\pi(2x \bmod 1)) \\ &= 2(\cos^2(\pi(2x \bmod 1)) - \sin^2(\pi(2x \bmod 1))) \\ &= 2(\cos^2(2\pi x) - \sin^2(2\pi x)) \\ &= 2(\cos^2(2\pi x) - (1 - \cos^2(2\pi x))) \\ &= 2(2 \cos^2(2\pi x) - 1) \\ &= 4 \cos^2(2\pi x) - 2 \end{aligned}$$

■

Again, the key fact about semi-conjugacy is that  $\varphi$  maps orbits of  $d$  to orbits of  $f$ . Thus, since  $d$  has a dense orbit  $f$  too has a dense orbit. Here's a concrete example illustrating this idea.

**Example 2.42** Find a point of period 11 for the chaotic quadratic  $f(x) = x^2 - 2$ .

**Solution.** First, it's easy to find an orbit of period 11 for the doubling map. One example is

$$0_2 00000000001 = \sum_{k=1}^{\infty} \frac{1}{2^{11k}} = \frac{1}{2047}.$$

The point behind conjugacy is that  $\varphi(1/2047) = 2 \cos(2\pi/2047)$  will be a point of period 11 for  $f$ . The reader is advised to check this numerically! □

## 2.9 A closer look at the bifurcation diagram

The bifurcation diagram of figure [Figure 2.26](#) is truly amazing.

## 2.10 Tent Maps

## 2.11 A few notes on computation

Many of the results in these notes have been illustrated on the computer and some of the exercises require a computational approach. Whenever using the computer, it is always wise to examine the results critically. Here's a simple numerical example where things clearly go awry. In it, we are iterating the function  $f(x) = x^2 - 9.1x + 1$  from the fixed point  $x_0 = 0.1$ . We should generate just a constant sequence.

```
x = 0.1
for i in range(20):
    x = x**2 - 9.1*x + 1
    print(x)
```

```
# Output:
0.09999999999999998
0.10000000000000002
0.09999999999999982
0.100000000000001608
0.099999999999985698
0.100000000000127285
0.09999999999886717
0.100000000010082188
0.0999999991026852
0.10000000798610176
0.09999992892369436
0.10000063257912528
0.09999437004618517
0.1000501066206484
0.09955405358690272
0.10396912194476915
0.06469056862056699
0.41550069522129274
-2.608415498784386
31.540412453236513
```

Uh-oh!

It must be understood that this is a simple consequence of the nature of floating point arithmetic. Part of the issue is that the decimal number 0.1 or  $1/10$  is not exactly representable in binary. In fact,

$$\frac{1}{10} = 0_2\overline{00011} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{3}{16^k}.$$

Thus, the computer *must* introduce round-off error in the computation. Furthermore, 0.1 is a repelling fixed point of the function. Thus, that round-off error is magnified with each iteration. Our study of dynamics has illuminated a critical issue in numerical computation!

In the terminology of numerical analysis, computation near an attractive fixed point is stable while computation near a repelling fixed point is unstable. Generally speaking, stable computation is trustworthy while unstable computation is not. The implication for the pictures that we see here is that illustration of attractive behavior should be just fine. In [Figure 2.9](#), for example, images (b) and (c) illustrate attraction to a fixed point that not only involves stable

computation but also agrees with our theoretical development. We are happy with those figures. The cobweb plot shown in [Figure 2.9](#) (d), however, should frankly be viewed with some suspicion.

The same can be said for the bifurcation diagram in [Figure 2.26](#). In much of that image, we see a gray smear indicating chaos. How can we trust that? Well, first, theory tells us that there really *is* chaos. That is, there are orbits that are dense in some interval for many  $c$  values. Furthermore, much of the image shows attractive regions and we can be confident in that portion.

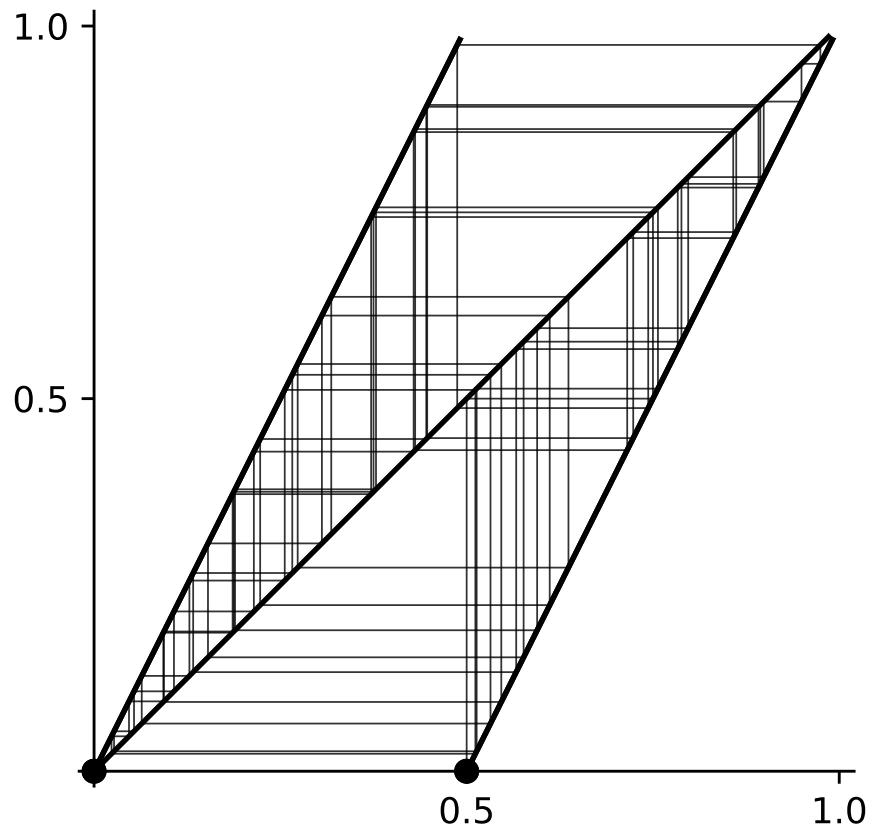
In fact, in many of the images that we will generate later - Julia sets, the Mandelbrot set, and similar images - the stable region dominates. Thus, we can be confident in overall image because the unstable region is the complement of the stable region. We might not be confident in computations involving some particular point, but we can be confident in the overall picture. (This will, perhaps, be more clear as we move into complex dynamics.)

Nonetheless, sometimes we want to experiment with genuinely unstable dynamics. One way to improve our confidence in these kinds of computations is to use high precision numbers. Consider, for example, the cobweb plot of the doubling map shown in [Figure 2.34](#). A naive approach to generate the first few terms of an orbit associated with the doubling map might be as follows:

```
import numpy as np
x = 1/np.pi
for i in range(55):
    x = 2*x%1
    print(x)
```

```
# Truncated output
0.636619772368
0.273239544735
0.54647908947
...
0.375
0.75
0.5
0.0
0.0
```

We've reached the fixed point zero and now we're stuck! Even if we iterate 1000 times, we'll generate a cobweb plot that looks like [Figure 2.43](#)



**Figure 2.43** An inaccurate cobweb plot

The cobweb plot shown in [Figure 2.34](#) was generated using the mpmath multi-precision library for Python with code that looked something like so:

```
from mpmath import mp
mp.prec = 1000
x = 1/mp.pi
for i in range(1000):
    x = 2*x%1
    print(float(x))
```

```
# Truncated output
0.6366197723675813
0.27323954473516265
0.5464790894703253
...
0.375
0.75
0.5
0.0
0.0
```

While the truncated output looks the same, note that this was after 1000 iterates. This behavior makes perfect sense if you understand that the doubling map loses one bit of precision with every iterate.

## 2.12 Exercises

1. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be continuously differentiable. We say that  $x_0$  is a simple root of  $f$  if  $f(x_0) = 0$  and  $f'(x_0) \neq 0$ . Show that if  $x_0$  is a simple root of  $f$ , then  $x_0$  is a super-attracting fixed point of the Newton's method iteration function  $N$  for  $f$ .
2. Find an example of a continuously differentiable function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that attracts no critical point.  
**Hint.** Draw a graph. Of course, you can't violate theorem [Theorem 2.25](#).
3. Let  $f(x) = x^2 - 4x + 5$ . Show that  $f$  has a super-attractive orbit of period 2.
4. Let  $f(x) = 3x^2 - 6x + 3.415$ . Find all attractive orbits of  $f$ .
5. Find a value of  $c$  such that  $f_c(x) = x^2 + c$  is affinely conjugate to  $g(x) = (x - 1)(x + 2)$ . Show that both functions have neutral fixed points.
6. Find an orbit of period 11 for the function  $g(x) = 4x(1 - x)$ .
7. We wish to find a number  $x_0 \in H = [0, 1)$  whose orbit is dense in  $H$  under iteration of  $g(x) = 4x(1 - x)$ .
  - (a) Outline a strategy for finding  $x_0$ .
  - (b) Find a decimal approximation to  $x_0$  that is valid to 10 decimal places.

# Chapter 3

## Self-similarity

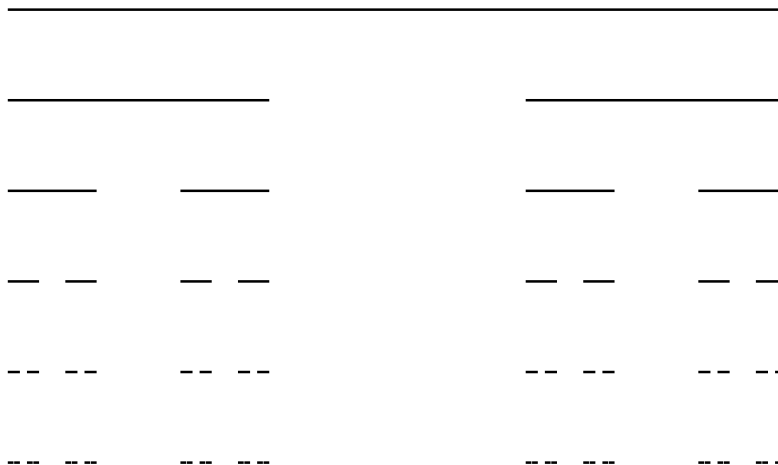
**Introduction.** At the end of [Chapter 2](#), we met a strange set called the Cantor Set. In this chapter, we'll see that the Cantor set is the prototypical member of a broad class of sets called the self-similar sets. Intuitively, a self-similar set is one that is composed of smaller copies of itself. To prepare for our exploration of these sets, we'll begin with another look at the Cantor set itself.

### 3.1 Another look at the Cantor set

Cantor constructed his set in the 1880's to help him understand a problem in Fourier series. While the set seemed unnatural to mathematicians of the time, it has become a central example in real analysis. Cantor's construction is as follows. Start with the unit interval  $I = [0, 1]$ , the set of all real numbers between 0 and 1 inclusive. Remove the open middle third  $(\frac{1}{3}, \frac{2}{3})$  of the interval  $I$  to obtain the two intervals  $I_1 = [0, \frac{1}{3}]$  and  $I_2 = [\frac{2}{3}, 1]$ . Then remove the open middle thirds of the intervals  $I_1$  and  $I_2$  to obtain the intervals  $I_{1,1} = [0, \frac{1}{9}]$ ,  $I_{1,2} = [\frac{2}{9}, \frac{1}{3}]$ ,  $I_{2,1} = [\frac{2}{3}, \frac{7}{9}]$ , and  $I_{2,2} = [\frac{8}{9}, 1]$ . Repeating this process inductively, we obtain  $2^n$  intervals of length  $1/3^n$  at the  $n^{\text{th}}$  stage. The cantor set  $C$  consists of all those points in  $I$  which are never removed at any stage. More precisely, if  $C_n$  denotes the union of all of the intervals left after the  $n^{\text{th}}$  stage of the construction, then

$$C = \bigcap_{n=1}^{\infty} C_n.$$

This process is illustrated in [Figure 3.1](#)



**Figure 3.1** Construction of the cantor set

It's clear that  $C$  should be self-similar, since the effect of the construction on the intervals  $I_1$  and  $I_2$  is the same as the effect on the whole interval  $I$ , but on a smaller scale. Thus  $C$  consists of two copies of itself scaled by the factor  $1/3$ .

The Cantor set has many non-intuitive properties. In some sense, it seems very small; if we were to assign a "length" to it, that length would have to be zero. Indeed, by its very construction it is contained in  $2^n$  intervals of length  $1/3^n$ . Thus the length of  $C_n$  is  $2^n/3^n$  which tends to zero as  $n \rightarrow \infty$ . Since  $C$  is contained in  $C_n$  for all  $n$ , the length of  $C$  must be zero. It might even appear that there is nothing left in  $C$  after tossing so much out of the original interval  $I$ . In reality, the Cantor set is a very rich set with infinitely many points. Recall that only open intervals are removed during the construction. Thus all of the infinitely many endpoints remain. For example,  $1/3$ ,  $2/3$ , and  $80/81$  are all in  $C$ . There are still many more points in  $C$ , however.

There is a general technique for finding points of the Cantor set. The first stage in the construction consists of the two intervals  $I_1$  and  $I_2$ . Choose one and discard the other. Now the interval we chose, say  $I_1$  for concreteness, contains two disjoint intervals,  $I_{1,1}$  and  $I_{1,2}$ , in the next stage of the construction. Choose one of those and discard the other. If we continue this process inductively, we obtain a nested sequence of closed intervals which will collapse down to a point in the Cantor set. For example, we might have chosen the interval  $I_1$  at the first stage. Then we could have chosen the interval  $I_{1,2}$  at the next stage. We might then choose to alternate between the first or second sub-interval at any point generating intervals of the form  $I_{1,2,1,2,\dots,1,2}$ . These intervals collapse down to a single point which is not the endpoint of any removed interval.

The process for finding points in  $C$  constructs a one to one correspondence with the set of infinite sequences of 1s and 2s. The sequence corresponding to a particular point in  $C$  might be called the address of that point. As we will see, this addressing scheme can be generalized to other situations and provides a powerful tool for understanding self-similar sets. Note, for example, that the addressing scheme implies that the Cantor set is uncountable.

A major question that we will address later in the book asks, "What is the dimension of the Cantor set?" Certainly, it is too small to be considered a one dimensional set; it is just a scattering of points along the unit interval with length zero. It is uncountable, however; perhaps it is too large to be considered

as zero dimensional. We will develop a notion of ‘‘fractal dimension’’ that quantitatively captures this in-betweenness.