Exam 1 type problems

For complex variables

Our first exam is next Wednesday and Friday. Topicwise:

- You should certainly know Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, as well as it's basic generalizations.
- You should know the implications of Euler's formula for complex multiplication.
- You should be able to break a function into it's real and imaginary parts e.g. If $f(z) = 1 + z + z^3$, then $f(x + iy) = u(x, y) + iv(x, y)$.
- You should know the Cauchy Riemann equations and be able to verify, for example, that u and v obtained from the previous problem satisfy them.
- You should be able to write down the cross-ratio - it's not so hard, if you understand what it's supposed to do.
- You should be able to use the cross-ratio to find some Mobius transformations in simple cases. For example, find the Mobius transformation mapping $1 \to 0$, $0 \to 1$ and $2 \to \infty$ or, maybe, $2 \to 3$, $3 \to 4$, $4 \to \infty$.

A few concrete definitions:

- Holomorphic function
- Euler's formula
- Mobius transformation
- The extended complex plane or Riemann sphere (pages 35-6)
- The complex exponential function (page 41)

Problemwise:

1. Use the definition of the derivative to show that the function $f(z) = |z|$ is nowhere differentiable.
2. Writing $z = x + iy$ where $x, y \in \mathbb{R}$, separate e^z into its real and imaginary parts.
3. Writing $f(x + iy) = u(x, y) + iv(x, y)$ where $x, y \in \mathbb{R}$ and u, v map $\mathbb{R}^2 \to \mathbb{R}$, use the Cauchy-Riemann equations to determine which of the following defines a holomorphic map.
   (a) $f(x + iy) = \left(x^2 - y^2 + e^x \cos(y)\right) + i\left(2xy + e^x \sin(y)\right)$
   (b) $f(x + iy) = (x + y) + i(x - y)$
   (c) $f(z) = \bar{z}$

4. Express the following complex numbers in the form $a + bi$.
   (a) $1/(1+i)$.
   (b) $\left(\frac{1+i}{\sqrt{2}}\right)^{100}$
5. Recall that the *cross-ratio* of four complex numbers z, z_1, z_2, z_3 is defined by

$$[z, z_1, z_2, z_3] = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}.$$

   (a) If $T(z) = [z, z_1, z_2, z_3]$, then what are the images of z_1, z_2, and $z_3$under T?
   (b) Use the cross-ratio to find a Mobius transformation that fixes 1, sends $0 \to -i$ and sends $\infty \to i$.
   (c) What is the image of the real line under the Mobius transformation you found in part (b)?
6. Let $R = \{z \in \mathbb{C} : 1 < |z| < 2, 0 \leq \arg(z) < \pi\}$ and let R^2 denote the image of R under the square function.
   (a) Sketch R in the plane. Be sure to indicate any edges *not* contained in R with dashed lines, while edges that *are* contained in R should be solid.
   (b) Is R open, closed, or neither? (You needn't prove or even justify your assertion.)
   (c) Sketch R^2 in the plane. Be sure to indicate the image of each edge of R
   (d) Is R^2 open, closed, or neither? (You needn't prove or even justify your assertion.)

Here are some problems from the text worth looking at:

Differentiability - chapter 2 11, 14, 15, 16

Mobius transformations - chapter 3 5, 7, 8, 9, 12, 14, 15, 18

Other functions - chapter 3 23, 26, 33

Theoretical - chapter 3 6