Claim: Suppose $S \subset \mathbb{R}$ is non-empty and bounded above. Let $2S = \{2x \in \mathbb{R} : x \in S\}$. Then $\sup(2S) = 2\sup(S)$.

Proof

Let $\beta = \sup(S)$. We must show that $\sup(2S) = 2\beta$.

Sub-claim 1: 2β is an upper bound for $2S$.

Sub-proof: Let $y \in 2S$. By def, there is an $x \in S$ such that $y = 2x$. For this x, we have $x \leq \beta$ so that $2x \leq 2\beta$ or $y \leq 2\beta$. Thus, 2β is an upper bound for $2S$.

Sub-claim 2: 2β is the least such upper bound.

Sub-proof: Suppose M is any upper bound for $2S$. Thus, $M \geq y$ for all $y \in 2S$. Equivalently, $M \geq 2x$ or $\frac{M}{2} \geq x$ for all $x \in S$. Thus $\frac{M}{2}$ is an upper bound for S and (since β is the least upper bound for S) $\beta \leq \frac{M}{2}$. Thus $2\beta \leq M$ so 2β is the least upper bound for $2S$.

Sub-claim 1 and sub-claim 2 together show that 2β is the sup of $2S$.