Real Analysis - Some final problems

1. Let $S = \{1/n^2 : n \in \mathbb{N}\}$. Use the definition of infimum to show that $\inf S = 0$.

2. Suppose that f and g are real functions on the open interval I that are both continuous at the point $c \in I$. Use the epsilon-delta definition of continuity to prove that $f+g$ is continuous at c.

3. Suppose that f and g are defined on an open interval containing the point c and that they are differentiable at c. Use the difference quotient to prove that

$$(af + bg)'(c) = af'(c) + bg'(c),$$

for any real numbers a and b.

4. Let

$$f_1(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   (a) Use the squeeze theorem to prove that $f_1(x)$ is continuous at zero.

   (b) Use the divergence criterion for functional limits (Corollary 4.2.5) to show that f_1 is not differentiable at zero.

5. Let

$$f_2(x) = \begin{cases} x^2\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and let $g(x) = x + 2f_2(x)$.

   (a) Compute $g'(x)$. Your answer should be a piecewise defined formula with a numeric value at zero and a formula for nonzero x.

   (b) Show that $g'(0) > 0$ but that g is not increasing in any neighborhood of zero.

6. Suppose that f is differentiable on an interval I and there is a number r in $(0, 1)$ with $|f'(x)| \leq r$ for all $x \in I$. Use the mean value theorem to prove that f is a contraction on I. Recall that f is called a contraction on I if there is a number r with $0 < r < 1$ and

$$|f(y) - f(x)| \leq r|y - x|$$

for all $x, y \in I$.

7. (Exercise 5.3.5): Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

8. (Corollary): Suppose that f is differentiable on $[0, 1]$ with $f(0) = -2$ and $f'(x) \geq 2$ for all $x \in [0, 1]$. Show that f has a unique fixed point.