Local theory of periodic orbits

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When we speak of *local* behavior''s, we mean the behavior of a complex analytic function near some point. In this set of notes, we examine the behavior of a complex analytic function near a fixed point or, by extension, near the points in a periodic orbit.

The simplest periodic orbit is that of an attractive fixed point. Note that this excludes the super-attractive case. While the behavior near a super-attractive fixed point has some qualitative similarities to the behavior near a fixed point, there are important differences and the analysis is somewhat different.

On the other hand, once we understand the behavior of an attractive fixed point, it's quite easy to extend that understanding to attractive orbits by examining the function $F = f^n$. We'll also be able to use our analysis of attractive fixed points to understand repelling fixed points by considering f^{-1}.

The most subtle type of fixed point is a neutral fixed point, where $|f'(z_0)| = 1$. In this case, $f'(z_0) = e^{i\theta}$ and there are a variety of possibilities depending on the number theoretic properties of θ.

It's worth emphasizing that the results here apply to complex analytic functions in general - polynomials, rational functions, transcendental functions, whatever. Thus, we can use these results when exploring the iteration of a truly wide variety of functions.

1 Linearization near an attractive fixed point

The simplest type of periodic orbit is that of an attractive fixed point. That is, we have a point $z_0 \in \mathbb{C}$ and an analytic function $f$defined in a neighborhood of z_0 such that $f(z_0) = z_0$ and $0 < |f'(z_0)| < 1$. In this case, the function *looks* like the linear function $L(z) = \lambda z$, where $\lambda = f'(z_0)$. A simpler statement, whose proof follows immediately from the definition of the derivative is that f is contractive near z_0

Lemma 1.1. *Suppose that z_0 is an attractive fixed point of f. Then, there is an $r > 0$ such that f^n converges uniformly to the constant function $z \to z_0$ on the disk $D_r(z_0)$.*

Proof. Choose an α such that $|f'(z_0)| < \alpha < 1$ and an $r > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \alpha.$$

Then, since $f(z_0) = z_0$, we have

$$|f(z) - z_0| < \alpha |z - z_0|$$

and, by induction,

$$|f^n(z) - z_0| < \alpha^n |z - z_0|.$$

The result follows as the statment is for all $z \in D_r(z_0)$. □

While nice, we can make a much more precise statement. In particular, f is analytically conjugate to a linear function L. Note that the following theorem 1.2 is stated assuming that the fixed point is zero. The fact that this extends to any fixed point is essentially the content of exercise 6.1.

Theorem 1.2. *Suppose that f is analytic at zero with $f(0) = 0$ and $f'(0) = \lambda$, where $0 < |\lambda| < 1$. Then, there is a neighborhood U of zero and an analytic function $\varphi : U \to \mathbb{C}$ that conjugates $L(z) = \lambda z$ to f, i.e.*

$$L \circ \varphi(z) = \varphi \circ f(z).$$

This is a tremendously important theorem with several well known proofs. We present two.

1.1 Constructing the conjugation as a functional limit

The proof of 1.2 consists of two essential steps - (1) the construction of an approximation to the solution and (2) the proof that the construction converges as the approximation is refined. As such, the proof can be appreciated on multiple levels. From a computational perspective an understanding of the construction is essential in a couple of algorithms we'll develop for image generation.

Proof. We define $\varphi_n(z) = \lambda^{-n} f^n(z)$. Then,

$$\varphi_n \circ f = \lambda^{-n} f^{n+1} = \lambda \varphi_{n+1}.$$

Now, if we *assume* that the sequence φ_n converges to φ, then φ must satisfy

$$\varphi \circ f = \lambda \varphi = L \circ \varphi.$$

The tricky part is proving convergence. To this end, let's choose a number $r > 0$ small enough and a positive constant C big enough so that

$$|f(z) - \lambda z| \leq C|z|^2$$

whenever $|z| < r$. We can do this, of course, because the λz is the first term in the Taylor series for f. Thus, an application of the reverse triangle inequality yields

$$|f(z)| \leq |\lambda||z| + C|z|^2 \leq (|\lambda| + Cr)|z|.$$

Then, by induction,

$$|f^n(z)| \leq (|\lambda| + Cr)^n |z|.$$

Now, if r is small enough so that

$$\delta \equiv \frac{(|\lambda| + Cr)^2}{|\lambda|} < 1,$$

we have

$$\begin{aligned} |\varphi_{n+1}(z) - \varphi_n(z)| &= \left| \frac{f(f^n(z)) - \lambda f^n(z)}{\lambda^{n+1}} \right| \\ &\leq C \frac{|f^n(z)|^2}{|\lambda|^{n+1}} \leq C \frac{\delta^n |z|^2}{|\lambda|}. \end{aligned}$$

This implies that the sequence φ_n is uniformly Cauchy on the disk $\{z : |z| < r\}$ and, therefore, uniformly convergent. □

1.2 Constructing the conjugation as a power series

The idea behind our next proof is to *assume* that the conjugation can be written as a power series. Like our previous proof, we should prove that the process converges. We'll skip that step here as we already know that the conjugation exists. Thus, this construction could be viewed as another way to generate an approximation.

Proof. Since f is analytic at zero, it can be expanded in a Taylor series. The conditions $f(0) = 0$ and $f'(0) = \lambda$ give us a bit of information on the coefficients, but they're mostly arbitrary:

$$f(z) = \lambda z + \sum_{k=2}^{\infty} a_k z^k.$$

Now, we seek an *analytic* function φ as the conjugacy, thus we assume that it's given by a power series:

$$\varphi(z) = \sum_{k=0}^{\infty} c_k z^k.$$

We hope to choose the coefficients c_k such that

$$\varphi \circ f(z) = L \circ \varphi(z).$$

Now, the right hand side is pretty simple:

$$L(\varphi(z)) = \lambda \left(c_0 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots\right) = \lambda c_0 + \lambda c_1 z + \lambda c_2 z^2 + \lambda c_3 z^3 + \cdots.$$

The expression on the left is a bit trickier:

$$\begin{aligned}\varphi(f(z)) = c_0 \quad & + c_1 \left(\lambda z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right) \\ & + c_2 \left(\lambda z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right)^2 \\ & + c_3 \left(\lambda z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right)^3 \\ & + c_4 \left(\lambda z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots\right)^4 + \cdots\end{aligned}$$

Nonetheless, we can compare coefficients of powers of z to see what the c_ks *must* be in order to make the conjugacy work. The constant term on the right is λc_0, while on the left it's just c_0. The only way that can work is if $c_0 = 0$. The coefficient of z on both sides is λc_1. I guess that means that $c_1$can be any non-zero value. This just means that there are many possible choices for the conjugacy parametrized by the choice of c_1. We'll make the simplest possible choice for c_1, namely $c_1 = 1$. The coefficient of z^2 on the right is λc_2 while on the left it's $c_1 a_2 + c_2 \lambda^2$. Since we already know that $c_1 = 1$, this yields

$$c_2 = \frac{a_2}{\lambda - \lambda^2}.$$

In general, when we compare the coefficients of z^n, we obtain an equation involving $c_0, c_1, \ldots, c_n$ as well as some of the a_ks, which are known. Since $c_0, c_1, \ldots, c_{n-1}$ are already known, we can solve for c_n to obtain a recursive procedure to find *all* the c_ks. Pushing the computations we have so far one step further to illustrate the point, we find that the coefficient of z^3 on the right λc_3 while on the left it's $c_1 a_3 + 2 c_2 \lambda a_2 + c_3 \lambda^3$. Using the known values for c_1 and c_2, we get

$$c_3 = \left(a_3 + 2\lambda \frac{a_2^2}{\lambda - \lambda^2}\right) / \left(\lambda - \lambda^3\right).$$

In general, each c_k is given by

$$c_k = \frac{P_k(a_2, \ldots, a_k, c_2, \ldots, c_{k-1})}{\lambda^k - \lambda},$$

where each P_k is a polynomial in the variables a_j and c_j with positive coefficients. □

1.3 Illustrations

Figure 1.3 illustrates a linearizing conjugacy for $f(z) = z^2 - z/2$. The red point is the origin. The curves emanating out of that point map to rays of constant argument under φ; the loops around that point (only one is explicit) map to circles of constant absolute value. There are many other points that have the same general appearance as the origin; those are pre-images of the origin under f or f^n for some n. Put another way, those points eventually map onto the origin under iteration of f. As we move away from the origin, the curves congregate on the Julia set of f. Outside of the Julia set appears blank as that's outside the domain of φ.

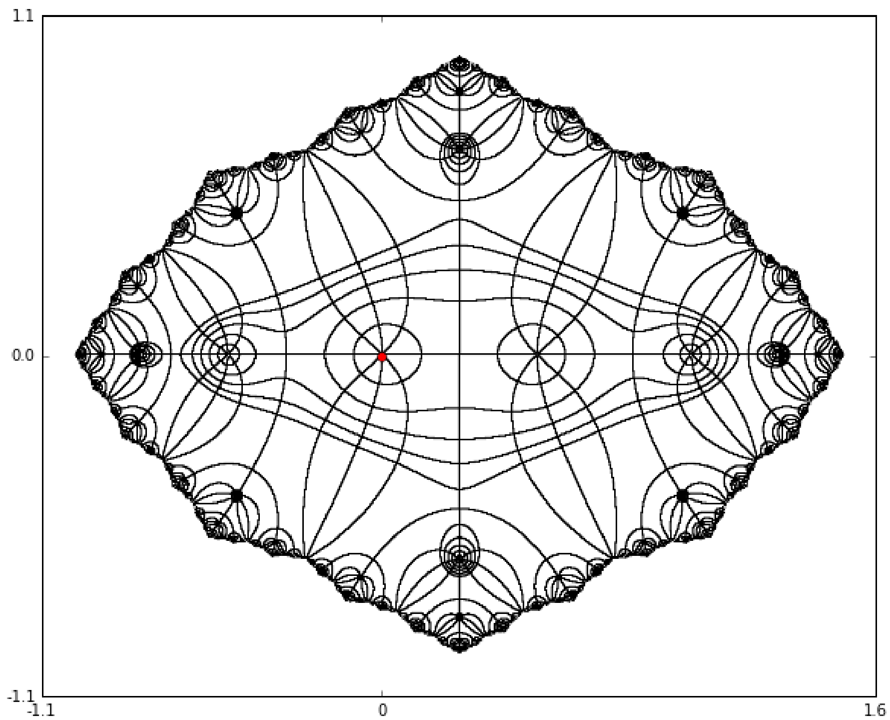

Figure 1.3: The conjugacy for $f(z) = z^2 - z/2$

Figure 1.4 (a) illustrates a linearizing conjugacy for $f(z) = z^2 + 0.8e^{\pi i/4}z$. In addition, the figure shows an orbit that spirals into an attractive fixed point. Since $f'(0) = 0.8e^{\pi i/4}$, the angle rotation from one point to the next is close to $\pi/4$, though the first few deviate measurably from that due to the z^2 term. Figure 1.4 (b) plots the image of those points under φ on a polar grid. As expected, they follow an orbit under iteration of the linear function $L(z) = 0.8e^{\pi i/4}z$.

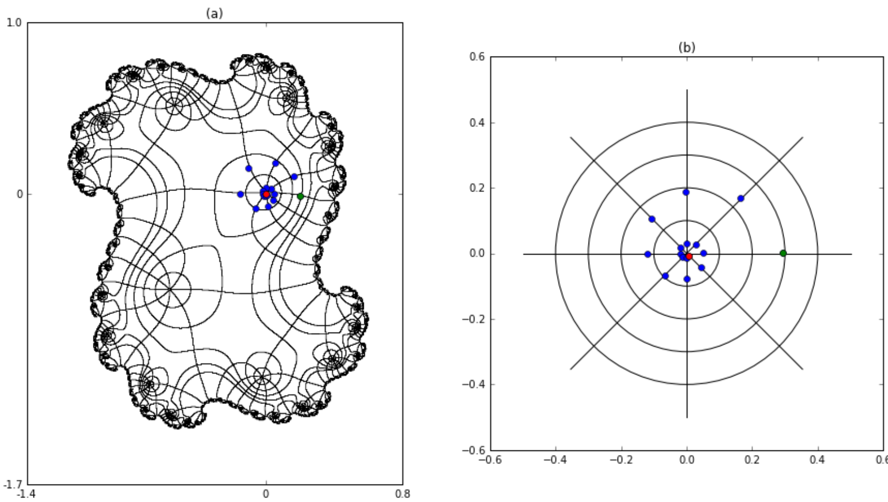

Figure 1.4: The conjugacy for $f(z) = z^2 + 0.8e^{\pi i/4}z$

2 Linearization near a repelling fixed point

3 Conjugation near a super-attractive fixed point

We now address the super attracting case. While there are some superficial similarities in the dynamics between the an attracting fixed point and a super-attracting fixed points, there are also substantial differences - starting with the analytical description of f. If z_0 is a super-attracting fixed point of f, then (expanding f in a Taylor series about z_0) we have

$$f(z) = z_0 + a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots$$

for some integer $m \geq 2$. It's immediately apparent that f is an m-to-1 function for every small neighborhood of z_0.

It will simplify matters a little bit to suppose that the fixed point is zero and that $a_m = 1$. The fact that we can do so without loss of generality is established in exercises 6.1 and 6.2. At the simplest level, a super-attractive fixed point is contained in a neighborhood where all the points converge to that fixed point under iteration; this is analogous to lemma 1.1

Lemma 3.1. *Suppose that f can be written as*

$$f(z) = z^m + a_{m+1}z^{m+1} + \cdots$$

in a neighborhood of the origin, so that zero is a super-attracting fixed point of order m for f. Then, there is a neighborhood of the origin where all the points of converge to zero under iteration of f.

Proof. By Taylor's theorem, there are positive constants C and r such that

$$|f(z)| < C|z|^m$$

for all z such that $|z| < r$. By induction, for such z

$$|f^n(z)| < C|z|^{m^n}.$$

The result follows. □

Theorem 3.2. *Suppose that f can be written as*

$$f(z) = z^m + a_{m+1} z^{m+1} + \cdots$$

in a neighborhood of the origin, so that zero is a super-attracting fixed point of order m for f. Then, f is analytically conjugate to $g(z) = z^m$.

Like theorem 1.2, this theorem can be appreciated on multiple levels. For the time being, we satisfy ourselves with the construction.

Proof. Let

$$\varphi_n(z) = f^n(z)^{m^{-n}},$$

which, by the inequalities in the proof of lemma 3.1, is well defined in some neighborhood of the origin. Note that

$$\varphi_{n-1} \circ f = (f^{n-1} \circ f)^{m^{-(n-1)}} = g \circ \varphi_n.$$

Thus, if φ_n converges to φ, then $\varphi \circ f = g \circ \varphi$. □

Figure 3.3 illustrates a this conjugacy for $f(z) = z^2 - z^4/8$. Note that the origin is a super-attracting fixed point - but not the only one. In fact, this map was created expanding $(z^2 - 1)^2 - 1$ and then conjugating to force the coefficient of z^2 to be one. Note also that the conjugacy extends only through the immediate basin of attraction of the origin. The curves that we see in side that basin are the inverse images of rays of constant argument.

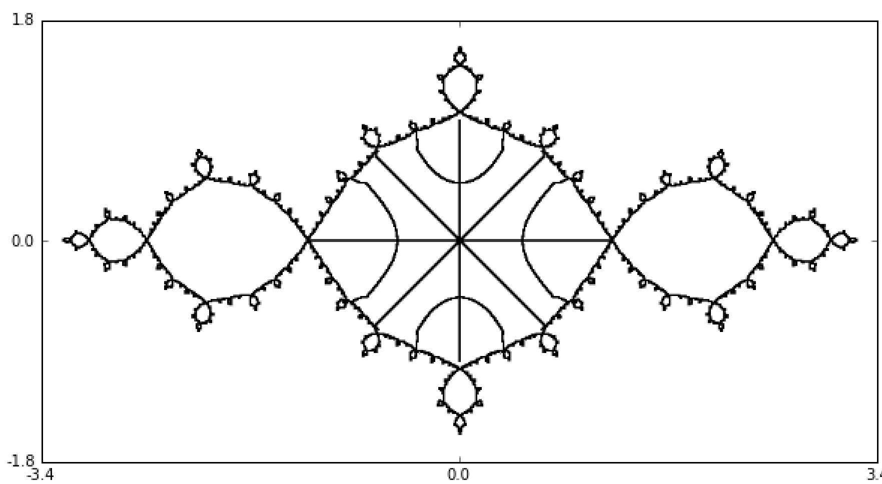

Figure 3.3: The conjugacy for $f(z) = z^2 - z^4/8$

4 Neutral points

The dynamics near a neutral point are considerably more complicated than the other cases. The dynamics depend quite subtely on the number theoretic properties of the mulitplier. For simplicity, we focus on the case where the origin is a neutral fixed point. This is really no restriction. If we have an orbit of degree $n > 1$, rather than a fixed point, then the points of that orbit are neutral fixed points of f^n. If the fixed point is not zero, then we can shift that fixed point via conjugation. Thus, we are interested, in this section, in functions of the form

$$f(z) = \lambda z + O(z^2),$$

where $|\lambda = 1|$.

There are a number of theorems that outline the possibilities that we will not prove, though we will use them to guide our exploration.

Theorem 4.1. *Suppose that f is a rational function with a neutral fixed point at z_0. Then f is linearizable at z_0 if and only if z_0 lies in the Fatou set of f.*

Theorem 4.2. *Suppose that f is a rational function with a neutral fixed point at z_0 that lies in some component of the Fatou set of f. Then, that component is simply connected and f is conjugate to a rotation of infinite order about z_0*

Theorem 4.2 implies immediately that for for a neutral point z_0 with multiplier $\lambda = e^{2\pi\alpha i}$ to lie in the Fatou set, the number α must be irrational. It can be proved that *most* irrational α (in the measure theoretic sense) work. One example is shown in figure 4.3. It would be instructive to play with that Julia set in a dynamic setting.

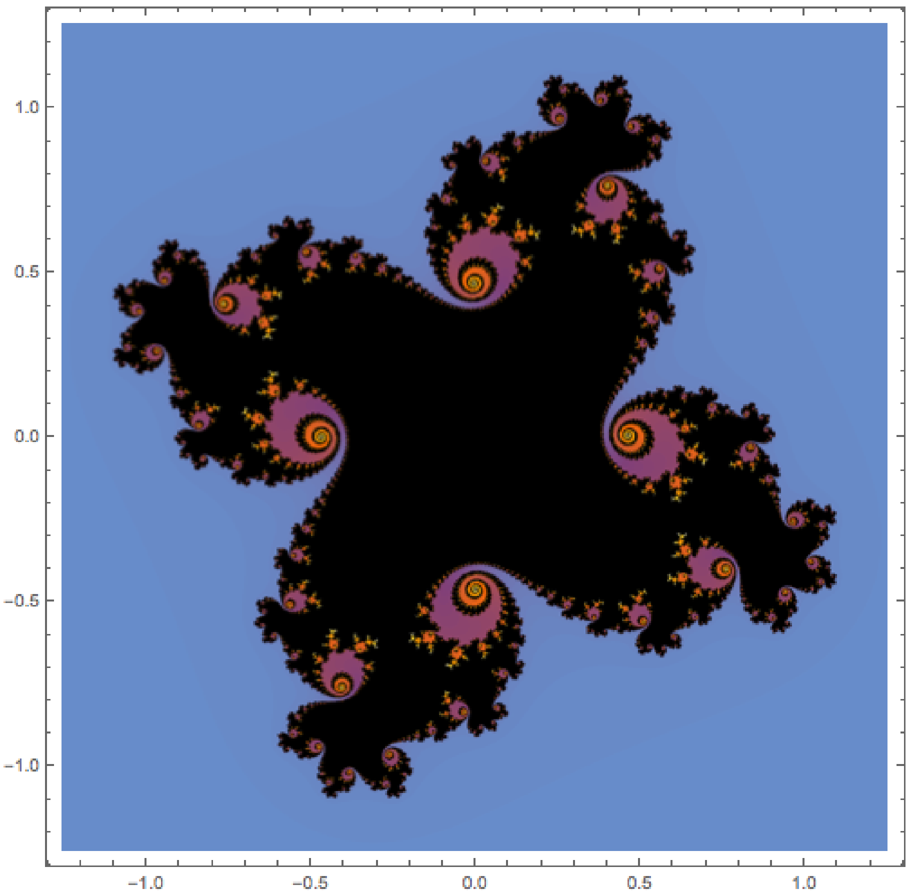

Figure 4.3: The Julia set of $f(z) = e^{\varphi i}z + z^5 \approx (-0.0472201 + 0.998885i)z + z^5$

It can also be proved that there are irrational numbers α and functions f with a fixed point z_0 with multiplier $\lambda = e^{2\pi\alpha i}$ such that z_0 lies in the Julia set of f. Such Julia sets are not well understood and at least some of them are known to be uncomputable in polynomial time. To the best of my knowledge, no useful images of such Julia sets have ever been produced.

4.1 Parabolic points

A parabolic fixed point is a fixed point with the property that the multiplier λ has the form $\lambda = e^{2\pi\alpha i}$ where α is rational number.

Theorem 4.4. *Suppose that f has the form*

$$f(z) = z + az^{n+1} + O(z^{n+2}).$$

Then zero is a fixed, parabolic point in the Fatou set of f. Suppose also that v is a solution of

$$nav^n = 1.$$

Then, f is repulsive in the direction v. Suppose, on the other hand that v is a solution of

$$nav^n = -1.$$

Then, f is attractive in the direction v.

5 Infinity as a super-attractive fixed point

Figure 5.1 illustrates the exterior conjugacy for $f(z) = z^2 - 1$.

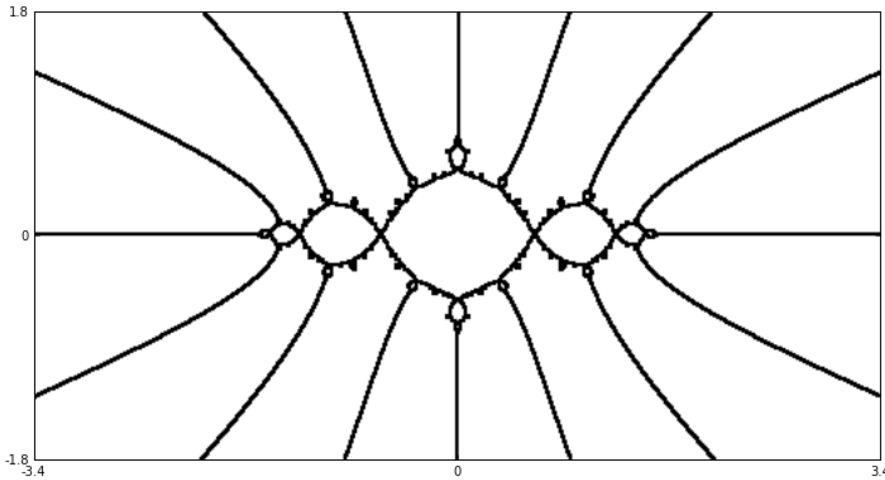

Figure 5.1: The exterior conjugacy for $f(z) = z^2 - 1$

Douady's rabbit is the Julia set for the function $f(z) = z^2 + c$ where $c \approx -0.122561 + 0.744862i$ is chosen to have a super-attractive orbit of period 3. Figure 5.2 (which was taken from WikiMedia Commons under the Creative Commons License shows Douady's rabbit, together with 3 external rays. It can be proven that these rays land at exactly the same point which, then, must be an articulation point.

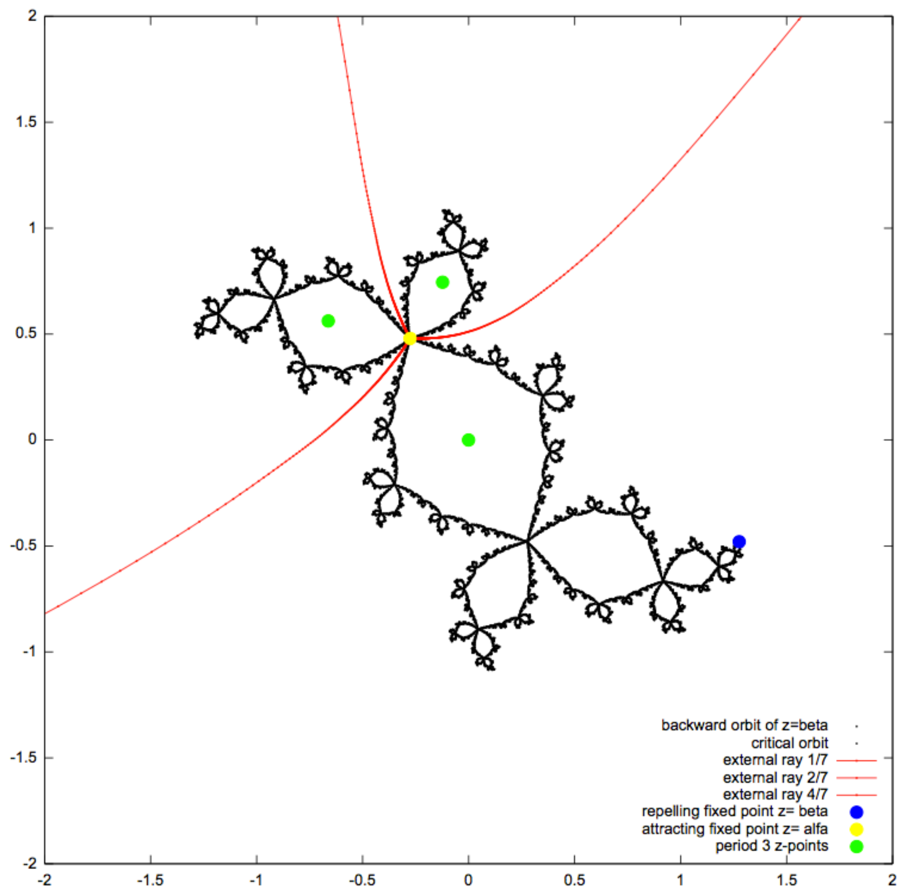

Figure 5.2: Three external rays for Douady's rabbit - from WikiMedia Commons

6 Exercises

1. Suppose that f has a fixed point at z_0. Show that the function g obtained by conjugating f with the function $\varphi(z) = z + z_0$ has a fixed point at zero. In addition, show that the conjugation preserves the nature of the fixed point as attractive, super-attractive, repulsive, or neutral.

2. Suppose that

$$f(z) = a_m z^m + a_{m+1} z^{m+1} + \cdots ,$$

so that f has a super-attractive fixed point of order m at zero. Show that the function g obtained by conjugating f with the function $\varphi(z) = a_m^{1/(m-1)} z$ conjugates f to a function g of the form

$$g(z) = z^m + a'_{m+1} z^{m+1} + \cdots$$